

# RESTRICTED UNIVERSES OF PARTIZAN MISÈRE GAMES

by

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DALHOUSIE UNIVERSITY

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*For Mr. King, who told me I would do something,  
and Dr. Gunther, who told me I should do math.*

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## Abstract

This thesis considers three restricted universes of partizan combinatorial games and finds new results for misère play using the recently-introduced theory of indistinguishability quotients.

The universes are defined by imposing three different conditions on game play: alternating, dicot (all-small), and dead-ending. General results are proved for each main universe, which in turn facilitate detailed analysis of specific subuniverses. In this way, misère monoids are constructed for alternating ends, for pairs of day-2 dicots, and for normal-play numbers, as well as for sets of positions that occur in variations of NIM, HACKENBUSH, and KAYLES, which fall into the alternating, dicot, and dead-ending universes, respectively.

Special attention is given to equivalency to zero in misère play. With a new sufficiency condition for the invertibility of games in a restricted universe, the thesis succeeds in demonstrating the invertibility (modulo specific universes) of all alternating ends, all but previous-win alternating non-ends, all but one day-2 dicot, over one thousand day-3 dicots, HACKENBUSH ‘sprigs’, dead ends, normal-play numbers, and partizan KAYLES positions.

Connections are drawn between the three universes, including the recurrence of monoids isomorphic to the group of integers under addition, and the similarities of universe-specific outcome determinants. Among the suggestions for future research is the further investigation of a natural and promising subset of dead-ending games called placement games.

## List of Symbols Used

$G^L$	The set of left options of a game $G$ .
$G^R$	The set of right options of a game $G$ .
$G^L$	A single left option of $G$ .
$G^R$	A single right option of $G$ .
$o^-(G)$	The outcome of $G$ under misère play.
$o^+(G)$	The outcome of $G$ under normal play.
$\overline{G}$	The <i>conjugate</i> of $G$ : $\overline{\{G^L \mid G^R\}} = \{G^R \mid G^L\}$ .
$cl(S)$	The <i>closure</i> of a set $S$ : all sums of positions in $S$ and their followers.
$\equiv$	Modular equivalence.
$\geq$	Modular inequality.
$\not\geq$	Modular strict inequality.
$\mathcal{A}$	The closure of alternating positions.
$\mathcal{A}_e$	The closure of alternating ends.
$\mathcal{P}$	The set of PENNY NIM positions.
$\mathcal{D}$	The set of dicot positions.
$\mathcal{S}$	The set of HACKENBUSH SPRIGS positions.
$\mathcal{E}$	The set of dead-ending positions.
$\mathcal{E}_e$	The closure of dead ends.
$\mathcal{K}$	The set of PARTIZAN KAYLES positions.
$A$	The alternating game $\{0 \mid \cdot\} = \mathbf{1}$ .
$B$	The alternating game $\{0, \overline{A} \mid \cdot\}$ .
$C$	The alternating game $\{\overline{B} \mid \cdot\}$ .
$D$	The alternating game $\{0, \overline{A} \mid 0, A\}$ .
$E$	The dicot game $\{0, * \mid *\}$ .
$*$	The dicot game <i>star</i> , $\{0 \mid 0\}$ .
$*_2$	The dicot game <i>star 2</i> , $\{0, * \mid 0, *\}$ .
$\uparrow$	The dicot game <i>up</i> , $\{0 \mid *\}$ .
$\uparrow*$	The dicot game <i>up star</i> , $\{0, * \mid 0\}$ .
$S_n$	A strip of $n$ squares (in KAYLES).

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# Chapter 1

## Introduction

*“I guess I’ll make it a spread misere,” said Dangerous Dan McGrew.*

— Robert W. Service, *The Shooting of Dan McGrew*

In various card games, a *misere* or *misère* bid is one in which the bidding player attempts to win as few tricks as possible. A player might make such a bid when dealt a particularly poor or ‘miserable’ hand. In combinatorial game theory, where games are won under *normal play* by the player who makes the last legal move, the term *misère* likewise means to ‘lose on purpose’: that is, the winner under *misère* play is the first player unable to move.

A combinatorial game is a two-player game of perfect information and no chance. Pure strategy games have been studied mathematically since the 1930s, when the Sprague-Grundy theory facilitated for the first time a general, abstract study of games [7, 20, 21] — or, at least, of normal-play games. In the 1970s, a complete theory of combinatorial games was presented by Conway in *On Numbers and Games* [5] and Berlekamp, Conway, and Guy in *Winning Ways* [4].

These theories develop the surprising and beautiful mathematical structure underlying normal-play games. For all games, there is a concept of addition, called the *disjunctive sum* of games, which arises naturally as the way in which two disjoint positions of a game are played side-by-side: on your turn, you can play in one position, or the other. There is a notion of *equality* of game positions, when two positions can be interchanged in a sum without affecting the outcome. And there is the *negative* of a game, where the roles (available moves) of the two players are swapped. These concepts combine to form the set of all normal-play games into an abelian group, complete with an additive identity called the *zero game*, in which neither player has a move.

The Fundamental Theorem of Combinatorial Games [1], which applies to both normal and *misère* play, says that every game position has exactly one of four possible

outcomes. With the two game players called ‘Left’ and ‘Right’, if there is a winning strategy for whichever player moves first, the position is *next-win*; if it is always winnable by whichever player moves second, it is *previous-win*; if Left can always win, whether she moves first or second, it is *left-win*; and finally, if Right can always win then it is *right-win*. One of the most wonderful properties of normal-play games is that every previous-win position is interchangeable with the zero game. Thus, for example, to prove that a game and its negative sum to zero, we need only find a winning strategy for the second player on the sum. The so-called *Tweedledum–Tweedledee* strategy does the trick: the second player symmetrically copies the first player’s moves and thereby gets the last move. Analysis of disjunctive sums in normal play is tremendously simplified by the fact that any previous-win position is irrelevant.

A game is called *impartial* if the legal moves at any time depend only on the game position and not on which of the players is to move next; in contrast, a non-impartial or *partizan* game may have a different set of moves available for Left than for Right. The Sprague-Grundy theorem states that every impartial game is equivalent under normal play to a position of the particular impartial game NIM<sup>1</sup>. For partizan games, a partial order is defined, which allows us to compare games in terms of their preference by Left or by Right, and games are assigned *values*, including numerical values, which in normal play interact with one another in precisely the same way as the underlying games. That is, the values preserve relationships of equality and inequality as well as operations of addition and negation, so that the negative of a game with value  $n$  has value  $-n$ , and so on. The algebra of normal-play games is thus elegant and practical.

### Changing the game

When the ending condition is switched from normal to misère — when we simply change the goal from getting the last move to avoiding the last move — everything falls apart. We can still add and negate games, can still compare games for equality and inequality, and can even still use the numerical values assigned to positions in normal play; however, most of these become virtually meaningless. Under misère play, *no* games are interchangeable with the zero game. In particular, the sum of

---

<sup>1</sup>Rule sets for NIM and all other games referenced in this thesis can be found in Appendix A.

a (non-zero) game and its negative is not equal to zero, and so there are no longer any inverses. The intuitive Tweedledum-Tweedledee strategy is now a terrible idea: if you copy your opponent's moves until the end of the game, you will get the last move, and lose! In fact, most intuition for the interaction of games in a sum is lost. It is no longer the case that every impartial game is a NIM position in disguise, and for partizan games, the numerical value system becomes practically useless, since, for example, the sum of a game of value  $n$  and a game of value  $m$  may not even be a number-valued game, let alone the game with value  $n + m$ .

For these reasons and others, misère games have been much less studied than normal-play games. One chapter of *On Numbers and Games* presents an analysis of 'How to Lose When You Must', and *Winning Ways* extends this work in their chapter 'Survival in the Lost World', but both texts consider only impartial misère games. The *genus theory* developed in the latter allowed for the analysis of certain impartial misère games, but left most unsolvable [16]. A theory for partizan misère games seemed, if possible, even more elusive.

### Learning to lose

Misère theory thus remained miserable until Thane Plambeck, along with Aaron Siegel, introduced a new way to consider equality of misère games [15, 17]. The original definition of equality, as given formally in Section 2.1, says that two games are equal if they can be interchanged in *any* sum of games without affecting the outcome of that sum. Plambeck suggested a weaker definition of equality, called *indistinguishability* or *equivalence*, whereby two game are equivalent *modulo*  $\mathcal{U}$  if they can be exchanged in any sum of games from  $\mathcal{U}$  without affecting the outcome. For example, we might take  $\mathcal{U}$  to be the set of all positions that occur in some particular game, such as DOMINEERING, and then two DOMINEERING positions are equivalent 'modulo DOMINEERING' if they are interchangeable in any sum of DOMINEERING positions. Although not as strong, theoretically, as the definition of equality, this is a natural and practical definition of equivalence, and its introduction has encouraged renewed interest in the study of misère games.

Given a set or *universe* of games  $\mathcal{U}$ , the equivalence relation described above forms a quotient semi-group of equivalence classes (with an identify but possibly

not inverses), which is called the *misère monoid* of  $\mathcal{U}$ . Although initially designed only for impartial games, the monoid construction works equally well for partizan games [19], and progress has been made in both camps over the past eight years. For partizan games, the subject of the present thesis, there have been new results for both general and restricted misère play. Section 2.4 summarizes the current status of general partizan misère theory, including work by Paul Ottaway, Aaron Siegel, and the other authors of the group G.A. Mesdal [10, 19].

Restricted partizan misère play, or partizan indistinguishability theory, has been mostly developed by the doctoral theses of Paul Ottaway [13] and Meghan Allen [2]. Although restricted equivalence is not explicitly employed in the former, Ottaway’s investigation of *consecutive move ban* (or *alternating*) games can be extended to describe the corresponding misère monoid. Allen computes and develops constructions for a number of finite misère monoids, and considers the invertibility of a specific position called *star* (where both players have exactly one move to zero) modulo various universes.

### **The next move**

It is clear that the exploration of partizan misère theory has just begun. There are many unanswered and unasked questions, which, with Plambeck’s indistinguishability theory, can now hope to be solved. What specific partizan games can we successfully analyze, by considering equivalence modulo only those positions that occur in the game? What are the misère monoids of various well-known sets of positions from normal-play, such as number-valued positions? In which universes and for which games do we find some resemblance of the algebraic properties from normal-play, such as invertibility?

This thesis answers the above questions and more. The first major result, presented at end of Chapter 2, gives a set of criteria for the invertibility of games in a restricted universe, which is used with great success in Chapters 3, 4, and 5. These three main chapters each consider a different restricted universe of partizan games, defined by the properties of alternating, dicot, and dead-ending games, respectively. Significant new results for misère play are developed in all three universes. Chapter 3 constructs the monoid of alternating ends, which has been published in a joint

paper with Richard Nowakowski and Paul Ottaway [11]. Chapter 4 includes a sufficiency condition for invertibility of dicot games and generalizes a well-known result of Meghan Allen. These results and the solution to HACKENBUSH SPRIGS, given in Section 4.5, appear in a joint paper with Neil McKay and Richard Nowakowski [9]. Among the main contributions of Chapter 5 is the monoid of all normal-play numbers, which is one of several results presented in a paper with Gabriel Renault [12].

Chapters 3, 4, and 5 each begin with definitions and general results for the given universe. This is followed by the analysis of various specific subuniverses. In particular, each chapter includes a complete solution to a game in its universe: variations of NIM, HACKENBUSH, and KAYLES, respectively. Finally, each chapter ends with a discussion of open questions and future research directions. Chapter 6 highlights the most promising of these future directions and discusses the overarching ideas that connect the three restricted universes.

The thesis begins with a chapter of background material, which provides the necessary terminology, notation, and prerequisite results for the discussions to follow.

## Chapter 2

### Prerequisite Material

#### 2.1 Basic Definitions

A *combinatorial game* is one of pure strategy, with no luck or chance, played by two-players ‘Left’ and ‘Right’ who have perfect information about the game. By convention, Left is a female who typically plays pieces that are blue or black, while Right is a male who generally plays red or white. The term *game* can refer to either a specific rule set, as in the ‘game of NIM’, or a single *position* of a game, such as ‘a game of NIM with one heap of two tokens’. A position  $G$  is defined by the positions to which Left and Right may legally move:  $G = \{G^{\mathbf{L}} \mid G^{\mathbf{R}}\}$ , where  $G^{\mathbf{L}} = \{G^{L_1}, G^{L_2}, \dots\}$  is the set of *left options* from  $G$  and  $G^{\mathbf{R}} = \{G^{R_1}, G^{R_2}, \dots\}$  is the set of *right options* from  $G$ . If the left and right options of a game are always the same, then the game is called *impartial*; otherwise, the game is called *partizan*.

The *game tree* of a position  $G$  is a downwards-directed tree, rooted at  $G$ , with branches to the left for each left option and branches to the right for each right option. Every vertex of the game tree represents a *follower* of  $G$ , as defined below.

**Definition 2.1.1.** *A game  $H$  is a follower of  $G$  if  $H$  can be reached from  $G$  by some sequence of (not necessarily alternating) moves. A proper follower is a follower that is not the original game itself.*

For the purposes of this thesis, a combinatorial game has a finite game tree, and games cannot end in a draw, so that one of the players is eventually declared the winner: under *normal play*, the first player unable to move loses, and under *misère play*, the first player unable to move wins. In both play conventions, the outcome classes *next* ( $\mathcal{N}$ ), *previous* ( $\mathcal{P}$ ), *left* ( $\mathcal{L}$ ), and *right* ( $\mathcal{R}$ ) are partially ordered as shown in Figure 2.1, with Left preferring moves towards the top and Right preferring moves towards the bottom. That is,  $\mathcal{L} > \mathcal{P} > \mathcal{R}$  and  $\mathcal{L} > \mathcal{N} > \mathcal{R}$ . To distinguish between the normal and misère outcomes of a game, the superscripts  $+$  and  $-$  are introduced:

$G \in \mathcal{N}^+$  means that  $G$  is next-win under normal play, while  $H \in \mathcal{L}^-$  means  $H$  is left-win under misère play. The outcome functions  $o^-(G)$  and  $o^+(G)$  are also used to identify the misère or normal outcome, respectively, of a game  $G$ .

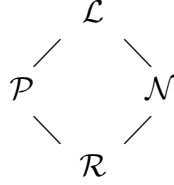


Figure 2.1: The partial order of outcome classes.

Many definitions from normal-play game theory<sup>1</sup> are used without modification for misère games, including disjunctive sum, equality, and inequality. These definitions are reviewed below, with the notation of the present thesis. A superscript  $+$  is used to indicate normal-play relations, while  $=$ ,  $\geq$ ,  $>$  without superscripts are used for misère-play relations.

**Definition 2.1.2.** *In normal and misère play, the sum of  $G$  and  $H$  is the game*

$$G + H = \{G^{\mathbf{L}} + H, G + H^{\mathbf{L}} \mid G^{\mathbf{R}} + H, G + H^{\mathbf{R}}\},$$

where  $G^{\mathbf{L}} + H$  is understood to mean the set of all sums  $G^{\mathbf{L}} + H$  for  $G^{\mathbf{L}} \in G^{\mathbf{L}}$ .

**Definition 2.1.3.** *The equality of two games in misère play is defined by*

$$G = H \text{ if and only if } o^-(G + X) = o^-(H + X) \text{ for all games } X;$$

*the equality of two games in normal play is defined by*

$$G =^+ H \text{ if and only if } o^+(G + X) = o^+(H + X) \text{ for all games } X.$$

**Definition 2.1.4.** *The inequality of two games in misère play is defined by*

$$G \geq H \text{ if and only if } o^-(G + X) \geq o^-(H + X) \text{ for all games } X,$$

$$G > H \text{ if and only if } G \geq H \text{ and } G \neq H;$$

*the inequality of two games in normal play is defined by*

$$G \geq^+ H \text{ if and only if } o^+(G + X) \geq o^+(H + X) \text{ for all games } X,$$

$$G >^+ H \text{ if and only if } G \geq^+ H \text{ and } G \neq^+ H.$$

---

<sup>1</sup>A complete overview of normal-play game theory can be found in [1].

Two positions with the same game tree are called *identical*; such games are also trivially *equal* in both normal and misère play, by Definition 2.1.3.

The definition of inequality leads to two game reductions: removing dominated options and bypassing reversible options. These reductions are well-known in normal play, and were relatively recently shown to also hold in misère play [10]. If  $G = \{G^{L_1}, G^{L_2}, \dots \mid G^R\}$  and  $G^{L_2} \geq G^{L_1}$ , then we say  $G^{L_2}$  *dominates*  $G^{L_1}$ , and in this case the game  $G$  is equal to the game with the dominated option removed, so that  $G = \{G^{L_2}, \dots \mid G^R\}$ . Dominated right options can similarly be removed from  $G^R$ : if  $G^{R_2} \leq G^{R_1}$  (that is, if  $G^{R_2}$  is at least as good for Right as  $G^{R_1}$ ) then  $\{G^L \mid G^{R_1}, G^{R_2}, \dots\} = \{G^L \mid G^{R_2}, \dots\}$ . A left option  $G^L$  is *reversible* if there is a right option  $G^{LR}$  of  $G^L$  such that  $G \geq G^{LR}$ , and in this case we can *bypass*  $G^L$ , so that  $G$  is equal to the game with  $G^L$  replaced by all the left options of  $G^{LR}$ :  $G = \{G^{LRL}, \dots \mid G^R\}$ . Again, reversible right options can likewise be bypassed. If  $G$  has no dominated or reversible options then  $G$  is in *canonical form*, as explicitly defined below. Uniqueness of canonical form is discussed in Section 2.4.

**Definition 2.1.5.** *The canonical form of a game  $G$  is the game  $H$  obtained from  $G$  by removing all dominated options and bypassing all reversible options.*

The height of the game tree of a position in canonical form is called the *birthday* of the game, with one game considered *simpler* than another if it has a smaller birthday. The simplest game is the *zero game*,  $0 = \{\cdot \mid \cdot\}$ , where the dot indicates an empty set of options. A game is said to be *born on day  $n$*  if its birthday is  $n$ ; so, for example,  $0$  is born on day 0, and the game  $\{0 \mid 0\}$ , called *star* and denoted  $*$ , is born on day 1.

In normal play, the *negative* of a game is defined recursively as  $-G = \{-G^R \mid -G^L\}$ , and is so-called because  $G + (-G) =^+ 0$  for all games  $G$ . Under misère play, however, this property holds only if  $G$  is identical to the zero game  $\{\cdot \mid \cdot\}$  [10]. To avoid confusion and inappropriate cancellation, we generally write  $\overline{G}$  instead of  $-G$  and refer to this game as the *conjugate* of  $G$ .

For normal-play games, there is an easy test of equality:  $G = 0$  if and only if  $G \in \mathcal{P}^+$ , and so  $G = H$  if and only if  $G - H \in \mathcal{P}^+$ . In misère play, no such test exists. Equality of misère games is difficult to prove and, moreover, is relatively rare: for example, besides  $\{\cdot \mid \cdot\}$  itself, there are no games equal to the zero game under misère play [10]. Plambeck [15] and Plambeck and Siegel [17] introduced a partial

solution to these challenges: redefine equality by restricting the game universe. This definition of modular equality (equivalence) and inequality is given below.

**Definition 2.1.6.** *For games  $G, H \in \mathcal{U}$ , the terms equivalence, inequality, and strict inequality modulo  $\mathcal{U}$  are defined by*

$$G \equiv H \pmod{\mathcal{U}} \text{ if and only if } o^-(G + X) = o^-(H + X) \text{ for all games } X \in \mathcal{U},$$

$$G \geq H \pmod{\mathcal{U}} \text{ if and only if } o^-(G + X) \geq o^-(H + X) \text{ for all games } X \in \mathcal{U},$$

$$G \not\equiv H \pmod{\mathcal{U}} \text{ if and only if } G \not\geq H \pmod{\mathcal{U}} \text{ and } G \not\leq H \pmod{\mathcal{U}}.$$

The words *equivalent* and *indistinguishable* are used interchangeably, and if  $G \not\equiv H \pmod{\mathcal{U}}$  then  $G$  and  $H$  are said to be *distinguishable*. In this case there must be a game  $X \in \mathcal{U}$  such that  $o^-(G + X) \neq o^-(H + X)$ , and we say that  $X$  *distinguishes*  $G$  and  $H$ .

As implied,  $\equiv$  is an equivalence relation: reflexivity, symmetry, and transitivity all follow trivially from the reflexivity, symmetry, and transitivity of the equality of outcomes. In fact, this definition of equivalence is a congruence relation, since  $H \equiv K \pmod{\mathcal{U}}$  implies that  $G + H \equiv G + K \pmod{\mathcal{U}}$ .

Given a universe  $\mathcal{U}$ , we can determine the equivalence classes under  $\equiv \pmod{\mathcal{U}}$  and form the quotient semi-group  $\mathcal{U}/\equiv$ . This quotient, together with the tetra-partition of elements into the sets  $\mathcal{P}^-$ ,  $\mathcal{N}^-$ ,  $\mathcal{R}^-$ , and  $\mathcal{L}^-$ , is called the *misère monoid* of the universe  $\mathcal{U}$ , denoted  $\mathcal{M}_{\mathcal{U}}$ . All universes in the present thesis are *closed under followers* in the sense that every follower of a position in the universe is also in the universe. Most, but not all, are *closed under conjugates*, meaning that  $G \in \mathcal{U}$  implies  $\overline{G} \in \mathcal{U}$ . If a set of games  $S$  is not closed under disjunctive sum then we usually consider the *closure* of the set,  $cl(S)$ , which is the set of all disjunctive sums of the games (and their followers).

In a restricted universe  $\mathcal{U}$ , a game  $G$  may satisfy  $G + \overline{G} \equiv 0 \pmod{\mathcal{U}}$ , and then  $G$  is said to be *invertible* modulo  $\mathcal{U}$ . It is an open question whether or not  $G + H \equiv 0 \pmod{\mathcal{U}}$  implies  $H \equiv \overline{G} \pmod{\mathcal{U}}$ , when  $\mathcal{U}$  is closed under conjugates. Section 5.6 shows that this implication fails for an ‘asymmetric’ universe: the set of PARTIZAN KAYLES positions has  $G + H \equiv 0$  and  $H \not\equiv \overline{G}$ , but in fact  $\overline{G}$  is not even in the universe. For the purposes of this thesis, the term *invertibility* will specifically refer to  $G$  and

$\overline{G}$ ; thus, a game  $G$  will be said to be *not invertible* modulo  $\mathcal{U}$  if it is shown that  $G + \overline{G} \not\equiv 0 \pmod{\mathcal{U}}$ , even though it is possible that  $G$  may have some other additive inverse in  $\mathcal{U}$ . We can partially justify this convention with the following conjecture.

**Conjecture 2.1.7.** *If  $\mathcal{U}$  is closed under conjugates then  $G + H \equiv 0 \pmod{\mathcal{U}}$  implies  $H \equiv \overline{G} \pmod{\mathcal{U}}$ .*

When a game  $H$  is invertible modulo  $\mathcal{U}$ , we have  $G \geq H$  if and only if  $G + \overline{H} \geq 0$ . To see this, note that  $G \geq H \pmod{\mathcal{U}}$  if and only if  $o^-(G + X) \geq o^-(H + X)$  for any game  $X \in \mathcal{U}$ . In particular, this holds if and only if  $o^-(G + \overline{H} + X) \geq o^-(H + \overline{H} + X) = o^-(X)$ ; that is, if and only if  $G + \overline{H} \geq 0$ .

Along with the basic background for misère analysis presented above, the following additional definitions are required. In Chapters 3 and 5, we encounter positions called *ends* (or *one-handed games*, as they are called in [13] and [11]).

**Definition 2.1.8.** *A left end is a position with no first move for Left (that is,  $G$  with  $G^L = \emptyset$ ), and a right end is a position with no first move for Right ( $G^R = \emptyset$ ). A game is an end if it is either a left end or a right end or both (the zero game).*

Positions where ends are forbidden — more precisely, where Left can move if and only if Right can move — are called *all-small* [5] in normal play and *dicot* games in misère. These games are the subject of Chapter 4 and are also discussed during the literature review in Section 2.4.

Further definitions specific to a particular universe are introduced as needed in the following chapters. Rule sets for common games, such as NIM and DOMINEERING, are included in Appendix A.

## 2.2 Notes on Notation

A discussion of notational conventions is necessary before we proceed. As implied above, for a given game  $G$ , we use  $G^L$  to denote a general left option and  $G^L$  to denote the set of all such options. We may refer to a single left option from the position  $G^R$  as  $G^{RL}$  and to the set of all left options from the position  $G^R$  as  $G^{RL}$ .

Arbitrary as well as particular games are denoted with uppercase Roman letters. Lowercase letters and juxtaposition are used for scalar multiplication, so that  $kG$

indicates the disjunctive sum of  $k$  copies of the game  $G$ . The Greek letter  $\delta$  is used when a multiple can only be 0 or 1. When a game has been shown to be invertible in a universe, the notation  $-G$  may be used interchangeably with  $\overline{G}$ . For example, a negative scalar multiple of  $G$  can be used to represent a multiple of  $\overline{G}$ : if  $k < 0$  then  $kG$  indicates  $|k|$  copies of  $\overline{G}$ .

Many ‘named’ games from normal play make appearances in the forthcoming discussions. Most often, the normal-play name and notation are used for a position that is identical to the normal-play canonical form of that game. For example, the game  $\{0 \mid 0\}$  is called *star* and denoted by  $*$ , and the game  $\{0 \mid *\}$  is called *up* and denoted by  $\uparrow$ . Other named positions are introduced as needed in subsequent chapters.

Normal-play *numbers* are defined and discussed in the next section. It is necessary to distinguish between the game  $\mathbf{n}$  with value  $n$  in normal play, and the number  $n$ , because, as shown below, these games do not behave as much (or at all) like numerical integers in misère play as they do in normal play. There is also potential for confusion between scalar multiples and games that are numbers. Thus, numbers and lowercase letters in bold print refer to positions that are identical to normal-play canonical forms of number-valued games. With the exception of Section 4.5 and Chapter 5, the zero game is excluded from this convention.

### 2.3 Normal-play Numbers

In normal play, an *integer* is a game  $\mathbf{n}$  whose canonical form is  $\{\mathbf{n} - \mathbf{1} \mid \cdot\}$ , where  $\mathbf{0} = 0 = \{\cdot \mid \cdot\}$ . A non-integer *number*  $\mathbf{a}$  is defined as

$$\mathbf{a} = \frac{\mathbf{m}}{\mathbf{2}^j} = \left\{ \frac{\mathbf{m} - \mathbf{1}}{\mathbf{2}^j} \mid \frac{\mathbf{m} + \mathbf{1}}{\mathbf{2}^j} \right\},$$

with  $j > 0$  and  $m$  odd. The set of all integer and non-integer (combinatorial game) numbers is thus the set of *dyadic rationals*, which we denote by  $\mathbb{Q}_2$ .

As mentioned in the previous section, the terms *integer* and *number* in misère play refer to games that are identical to canonical-form normal-play integers and numbers. This is done for convenience, and not because numbers are number-like in misère play — in general, they are not. For example, numbers are not totally ordered in general misère play, and so  $n > m$  does not imply  $\mathbf{n} > \mathbf{m}$ . Similarly, as mentioned in the opening chapter, the sum of two non-integer numbers  $\mathbf{n}$  and  $\mathbf{m}$

may not even be a number, let alone the game corresponding to the number  $n + m$ . Normal-play numbers make several appearances in this thesis, especially in Section 4.5 and Chapter 5. Accordingly, a few basic properties of these games are established here.

For a non-integer number  $\mathbf{a} = \{\mathbf{a}^L \mid \mathbf{a}^R\}$  in normal-play canonical form, the position  $\mathbf{a}$  is the simplest number such that  $a^L < a < a^R$ . Note that  $a^L \in \mathbb{Q}_2$  represents the numerical value of the left option of  $\mathbf{a}$  in normal-play canonical form. The following additional facts are required in Chapters 4 and 5. The first proposition shows that if  $\mathbf{a}$  is a non-integer number, then either  $\mathbf{a}^{LR}$  or  $\mathbf{a}^{RL}$  exists. In general we may not have both: for example, the game  $\mathbf{1}/\mathbf{2} = \{\mathbf{0} \mid \mathbf{1}\}$  has no Right response to the left option  $\mathbf{0}$ .

**Proposition 2.3.1.** *If  $a \in \mathbb{Q}_2 \setminus \mathbb{Z}$  then at least one of  $\mathbf{a}^{RL}$  and  $\mathbf{a}^{LR}$  exists, and either  $\mathbf{a}^L = \mathbf{a}^{RL}$  or  $\mathbf{a}^R = \mathbf{a}^{LR}$ .*

*Proof.* Let  $\mathbf{a} = \mathbf{m}/\mathbf{2}^j$  with  $j > 0$  and  $m$  odd. If  $m \equiv 1 \pmod{4}$  then

$$\mathbf{a}^L = \frac{\mathbf{m} - \mathbf{1}}{\mathbf{2}^j}, \quad \mathbf{a}^R = \frac{\mathbf{m} + \mathbf{1}}{\mathbf{2}^j} = \frac{\frac{\mathbf{m} + \mathbf{1}}{\mathbf{2}}}{\mathbf{2}^{j-1}} = \left\{ \frac{\frac{\mathbf{m} - \mathbf{1}}{\mathbf{2}}}{\mathbf{2}^{j-1}} \mid \frac{\frac{\mathbf{m} + \mathbf{3}}{\mathbf{4}}}{\mathbf{2}^{j-1}} \right\},$$

so  $\mathbf{a}^L = \mathbf{a}^{RL}$ . Otherwise,  $m \equiv 3 \pmod{4}$  and then

$$\mathbf{a}^L = \frac{\mathbf{m} - \mathbf{1}}{\mathbf{2}^j} = \frac{\frac{\mathbf{m} - \mathbf{1}}{\mathbf{2}}}{\mathbf{2}^{j-1}} = \left\{ \frac{\frac{\mathbf{m} - \mathbf{3}}{\mathbf{2}}}{\mathbf{2}^{j-1}} \mid \frac{\frac{\mathbf{m} + \mathbf{1}}{\mathbf{2}}}{\mathbf{2}^{j-1}} \right\}, \quad \mathbf{a}^R = \frac{\mathbf{m} + \mathbf{1}}{\mathbf{2}^j},$$

so  $\mathbf{a}^R = \mathbf{a}^{LR}$ . □

**Proposition 2.3.2.** *If  $a \in \mathbb{Q}_2 \setminus \mathbb{Z}$  then  $\mathbf{a}^{RL} < a$  and  $\mathbf{a}^{LR} > a$ , when those options exist.*

*Proof.* From Proposition 2.3.1, either  $\mathbf{a}^L = \mathbf{a}^{RL}$  or  $\mathbf{a}^R = \mathbf{a}^{LR}$ , and so either  $\mathbf{a}^L = \mathbf{a}^{RL}$  or  $\mathbf{a}^R = \mathbf{a}^{LR}$ . Since  $a^L < a < a^R$ , this gives  $\mathbf{a}^{RL} < a$  when  $\mathbf{a}^{RL} = \mathbf{a}^L$  and  $\mathbf{a}^{LR} > a$  when  $\mathbf{a}^{LR} = \mathbf{a}^R$ . It remains to show  $\mathbf{a}^{LR} > a$  when  $\mathbf{a}^{RL} = \mathbf{a}^L$  (that is, when  $m \equiv 1 \pmod{4}$ ), and the symmetric result, which can be omitted. But this follows from the proof of Proposition 2.3.1, since when  $m \equiv 1 \pmod{4}$ ,

$$\mathbf{a}^L = \frac{\mathbf{m} - \mathbf{1}}{\mathbf{2}^j} = \frac{\frac{\mathbf{m} - \mathbf{1}}{\mathbf{4}}}{\mathbf{2}^{j-2}} = \left\{ \frac{\frac{\mathbf{m} - \mathbf{5}}{\mathbf{4}}}{\mathbf{2}^{j-2}} \mid \frac{\frac{\mathbf{m} + \mathbf{3}}{\mathbf{4}}}{\mathbf{2}^{j-2}} \right\} = \left\{ \frac{\mathbf{m} - \mathbf{5}}{\mathbf{2}^j} \mid \frac{\mathbf{m} + \mathbf{3}}{\mathbf{2}^j} \right\},$$

and so the underlying numbers satisfy

$$a^{LR} = \frac{m+3}{2^j} > a = \frac{m}{2^j}.$$

□

## 2.4 General Results for Misère Play

This section outlines the properties of general misère play that are used or referenced in this thesis. Unlike most of the results of the following three chapters, these properties hold in the non-restricted universe of misère games, or in *any* restricted universe. The relevant results from the literature on misère games are reviewed first, followed by an original result which holds for any universe of misère games and which is subsequently applied to each of the specific universes of this thesis.

As mentioned in Section 2.1, in general misère play, the only game that satisfies  $\sigma^-(G + X) = \sigma^-(X)$  for all games  $X$  is the zero game  $G = \{\cdot \mid \cdot\}$  [10]. In particular, the position  $G + \overline{G}$  cannot equal 0 for any game  $G$  that is not identical to 0. Thus, the set of misère games has no non-zero inverses; in contrast, the set of normal-play games forms an group.

In some restricted universes we do find that all games, or particular subsets of games, are invertible. For example, Meghan Allen showed that  $* + * \equiv 0$  in any universe of dicot games [3]. She poses the problem of identifying other universes in which  $* + * \equiv 0$ , as a potential future direction of misère research. It is well-known that the equivalence cannot hold in any universe containing the position  $\{0 \mid \cdot\}$ . Two of the three universes considered in the present thesis contain this position; however, the dicot universe considered in Chapter 4 does not, and there the equivalence  $* + * \equiv 0$  (modulo dicot games) is generalized with a number of stronger results.

The authors of [10] establish several other results for general misère play; some of these help to explain “what makes misère play so miserable” [10], while others provide much encouragement for future research in misère games. On the miserable side, we have that no general relationship exists between the normal-play outcome and misère-play outcome of a game. That is, knowing that a position is left-win in normal-play tells us nothing about its outcome in misère: for each outcome class  $\mathcal{C}$ ,

there are games in  $\mathcal{L}^+$  and  $\mathcal{C}^-$ . The same is true for each of the other three outcomes besides  $\mathcal{L}^+$ . The fact that all possible combinations of normal and misère outcomes are attainable comes into play in Chapter 3.

We also know that nothing can be said about the addition table of outcome classes in misère play [10]. In normal play, the outcomes of  $G$  and  $H$  usually give some indication of the outcome of  $G + H$ ; for example, the sum of two left-win positions is always left-win. The other relationships are illustrated in Figure 2.2. In misère play, as indicated in the same figure, there are no such relationships: for any three (not necessarily distinct) outcomes  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ , there are positions known to satisfy  $G \in \mathcal{C}_1^-$ ,  $H \in \mathcal{C}_2^-$ , and  $G + H \in \mathcal{C}_3^-$ .

+	$\mathcal{N}^+$	$\mathcal{P}^+$	$\mathcal{L}^+$	$\mathcal{R}^+$
$\mathcal{N}^+$	?	$\mathcal{N}^+$	$\mathcal{N}^+ \cup \mathcal{L}^+$	$\mathcal{N}^+ \cup \mathcal{R}^+$
$\mathcal{P}^+$	$\mathcal{N}^+$	$\mathcal{P}^+$	$\mathcal{L}^+$	$\mathcal{R}^+$
$\mathcal{L}^+$	$\mathcal{N}^+ \cup \mathcal{L}^+$	$\mathcal{L}^+$	$\mathcal{L}^+$	?
$\mathcal{R}^+$	$\mathcal{N}^+ \cup \mathcal{R}^+$	$\mathcal{R}^+$	?	$\mathcal{R}^+$

+	$\mathcal{N}^-$	$\mathcal{P}^-$	$\mathcal{L}^-$	$\mathcal{R}^-$
$\mathcal{N}^-$	?	?	?	?
$\mathcal{P}^-$	?	?	?	?
$\mathcal{L}^-$	?	?	?	?
$\mathcal{R}^-$	?	?	?	?

Figure 2.2: The outcome of  $G + H$  given the outcome of  $G$  and  $H$ , in normal play (left) and misère play (right).

Fortunately, even with such loss of structure as described above, there are some techniques from normal-play analysis that work and are useful in misère play. One of these is the so-called *hand-tying* principle. In normal-play, this principle says that if two games  $G$  and  $H$  differ only by the addition of one or more extra left options to  $G$ , then Left can do at least as well playing  $G$  as playing  $H$ . That is,  $G \geq H$ . This is because, at worst, Left can ‘tie her hand’ and ignore the extra options, thereby essentially playing the game  $H$  instead of  $G$ . In misère play, the same argument holds, with one stipulation: the set  $H^{\mathbf{L}}$  of left options cannot be empty. If it is, adding a left option is not always beneficial to Left, who is trying to run out of moves before Right. However, when there already exists at least one left option, Left can simply ignore any additional ones. This idea is used many times in the present thesis, and so we record it here as a lemma.

**Lemma 2.4.1.** [10] *If  $G^{\mathbf{L}} \supseteq H^{\mathbf{L}}$  and  $G^{\mathbf{R}} = H^{\mathbf{R}}$ , with  $H^{\mathbf{L}} \neq \emptyset$ , then  $G \geq H$  in both normal and misère play. If  $G^{\mathbf{L}} = H^{\mathbf{L}}$  and  $G^{\mathbf{R}} \subseteq H^{\mathbf{R}}$ , with  $G^{\mathbf{R}} \neq \emptyset$ , then  $G \geq H$  in both normal and misère play.*

As stated in the lemma, the above arguments hold for Right as well as Left. Note that this inequality is true in general misère play, without restricting to any particular subuniverse.

The authors of [10] also showed that misère games, like normal-play games, can be simplified by removing dominated options and bypassing reversible options, as described in Section 2.1. Aaron Siegel further showed that the simplified game obtained by removing all dominated options and bypassing all reversible options is unique; that is, misère games exhibit unique canonical forms [19].

The following theorem is an original result which appears in the joint manuscript [12]. It is used repeatedly in the following chapters to demonstrate invertibility of a single game or a set of games. Initially, similar, universe-specific criteria were being used for this purpose; Theorem 2.4.2 is the underlying argument that was common to each. Recall that a universe  $\mathcal{U}$  is ‘closed under conjugates’ if  $\overline{G} \in \mathcal{U}$  for every  $G \in \mathcal{U}$ , and  $\mathcal{U}$  is ‘closed under followers’ if  $H \in \mathcal{U}$  for any follower  $H$  of every game  $G \in \mathcal{U}$ . Also recall from Definition 2.1.8 that a ‘left end’ is a position with no first move for Left.

**Theorem 2.4.2.** *Let  $U$  be any game universe closed under conjugates, and let  $S \subseteq U$  be a set of games closed under followers. If  $G + \overline{G} + X \in \mathcal{L}^- \cup \mathcal{N}^-$  for every game  $G \in S$  and every left end  $X \in \mathcal{U}$ , then  $G + \overline{G} \equiv 0 \pmod{\mathcal{U}}$  for every  $G \in S$ .*

*Proof.* Let  $S$  be a set of games with the given conditions. Since  $U$  is closed under conjugates, by symmetry we also have  $G + \overline{G} + X \in \mathcal{R}^- \cup \mathcal{N}^-$  for every  $G \in S$  and every right end  $X \in \mathcal{U}$ .

Let  $G$  be any game in  $S$  and let  $X$  be any left end in  $\mathcal{U}$ . Since  $S$  is closed under followers, we have  $H + \overline{H} + X \in \mathcal{L}^- \cup \mathcal{N}^-$  for every follower  $H$  of  $G$ ; assume inductively that  $H + \overline{H} \equiv 0 \pmod{\mathcal{U}}$  for every follower  $H$  of  $G$ . Let  $Y$  be any game in  $\mathcal{U}$ , and suppose Left wins  $Y$ . We must show that Left can win  $G + \overline{G} + Y$ .

Left should follow her usual strategy in  $Y$ ; if Right plays in  $G$  or  $\overline{G}$  to, say,  $G^R + \overline{G} + Y'$ , with  $Y' \in \mathcal{L}^- \cup \mathcal{P}^-$ , then Left copies his move and wins as the second player on  $G^R + \overline{G}^L + Y' = G^R + \overline{G}^R + Y' \equiv 0 + Y'$ , by induction. Otherwise, once Left runs out of moves in  $Y$ , say at a left end  $Y''$ , she wins playing next on  $G + \overline{G} + Y''$  by assumption. A symmetric argument shows that Right wins  $G + \overline{G} + Y$  whenever

he wins  $Y$ , and so  $o^-(G + \overline{G} + Y) = o^-(Y)$  for every  $Y \in \mathcal{U}$ . By definition, this gives  $G + \overline{G} \equiv 0 \pmod{\mathcal{U}}$ .  $\square$

The proof of Theorem 2.4.2 uses a technique that is common practice in misère analysis and which is used frequently in the present thesis. The main argument above begins with the phrase ‘suppose Left wins  $Y$ ’. This appears to be ambiguous, or incomplete; does Left win  $Y$  playing first or playing second? The implied assumption with such a statement is that the the argument to follow holds for both cases. For example, in the proof of Theorem 2.4.2, if Left wins  $Y$  playing first then she wins  $G + \overline{G} + Y$  playing first by making her good first move in  $Y$  and then following her strategy as usual, responding to Right playing in  $G$  or  $\overline{G}$  as described above. If Left wins  $Y$  playing second and Right plays first in  $G + \overline{G} + Y$ , then Left either follows her strategy as usual, as long as Right is playing in  $Y$ , or responds as described if Right plays in  $G + \overline{G}$ . It is clear that whether Left wins first or second has no affect on the argument of the proof, and when this is the case (as it almost always is), the phrase ‘Left wins’ will be used without clarification.

We are now ready to begin analysis of the first of three restricted universes of partizan misère games.

## Chapter 3

### The Alternating Universe

#### 3.1 Introduction to Alternating Games

Consider the variation<sup>1</sup> PENNY NIM of a single-heap game of NIM. The game begins with a stack of pennies that are either all heads-up or all tails-up. Left can play on a tails-up stack and Right can play on a heads-up stack, by removing at least one penny and then inverting any remaining coins. Under misère rules, the first player unable to move is the winner.

When several stacks are played as a disjunctive sum, this game has the property that neither player can make two consecutive moves in a single component. Such a restriction is very interesting for misère play. *Alternating* or *consecutive move ban* games were first studied by Paul Ottaway [13], as a set of misère games with restricted options. Definition 3.1.1 below establishes precisely what is meant by this particular restriction.

**Definition 3.1.1.** *A game  $G$  is alternating if  $G^{LL} = \emptyset$  and  $G^{RR} = \emptyset$  for all left and right options  $G^L, G^R$  of  $G$ , and if every follower of  $G$  is also alternating.*

In PENNY NIM, each component is an *end* (Definition 2.1.8); that is, only one player has an option from the initial position. In general, a non-end position can also be alternating. If we place a heads-up or tails-up stack of pennies on its side and allow either player to move by taking some pennies and orienting the stack appropriately, then PENNY NIM would have both end and non-end positions. Figure 3.1 shows an example of a disjunctive sum of this generalized version of the game, with a black-up stack of size three, a white-up stack of size two, a sideways stack of size three and a sideways stack of size one. Who wins this sum? The answer is given in Section 3.5, following a complete solution to the game of PENNY NIM.

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<sup>1</sup>This game was first introduced as one of several ‘coin-flipping games’ in [13].

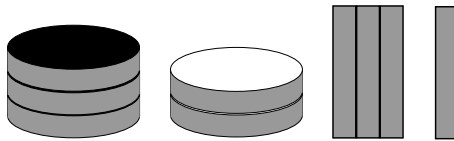


Figure 3.1: An example of the game PENNY NIM with sideways stacks.

Unlike the universes of Chapters 4 and 5, the set of all alternating positions is not closed under addition. For example, the normal-play integer  $\mathbf{1} = \{0 \mid \cdot\}$  is (trivially) alternating, since no player can make two consecutive moves, but the disjunctive sum  $\mathbf{1} + \mathbf{1} = \{\mathbf{1} \mid \cdot\}$  is not, since Left can make two moves in a row. To compensate, consider the *closure* or set of all disjunctive sums of alternating positions. This universe is denoted  $\mathcal{A}$ , and the subuniverse that is the closure of all alternating ends is denoted  $\mathcal{A}_e$ .

Alternating ends and non-ends are both studied in [13]; the contribution of the present chapter is to consider these games in the context of misère monoids and equivalency modulo a restricted universe. Section 3.2 begins by determining the equivalence classes of  $\mathcal{A}_e$  and the outcome of a general sum in this universe, thereby describing the misère monoid of the closure of alternating ends. This work, which appears in a joint publication with Richard Nowakowski and Paul Ottaway [11], is extended in Section 3.3, with the equivalence classes of individual non-end alternating positions. Although the entire monoid of  $\mathcal{A}$  has not been found, Section 3.4 makes an initial attempt at analyzing sums with non-end alternating positions. Finally, Section 3.5 applies many of the results of the preceding sections in order to solve the generalized version of PENNY NIM described above.

When analyzing alternating games, it is useful to classify a position by both its misère outcome and normal outcome. The outcome pair can be denoted by writing  $G \in o^-(G) \cap o^+(G)$ ; for example, the integer  $\mathbf{1}$  is in the outcome intersection  $\mathcal{L}^- \cap \mathcal{R}^+$ . In general, as discussed in Chapter 2, the normal-play outcome of a game has little or no relation to its misère-play outcome, and nor do the strategies for one ending condition have much significance when playing under the other. We will see that alternating games are among the few exceptions to this rule.

### 3.2 Alternating Ends

In normal play, every “one-sided” game  $\{G^L \mid \cdot\}$  or  $\{\cdot \mid G^R\}$  is equivalent to an integer. In misère play, by contrast, such games represent significant pathologies and are the source of much complication. [19]

Given this reflection on ends in general, it is a pleasant surprise that there is simple, elegant structure among *alternating* ends. In this section we establish the equivalence classes of alternating ends within the universe  $\mathcal{A}$  of all alternating games, and then narrow our scope to the universe  $\mathcal{A}_e$  of alternating ends alone, in order to compute the misère monoid  $\mathcal{M}_{\mathcal{A}_e}$ . The results for ends that also hold in the larger universe are crucial for the analysis of  $\mathcal{A}$  in Sections 3.3 and 3.4, since every option of a general alternating position is an alternating end.

We begin by sorting alternating ends according to both their misère and normal outcomes. As mentioned in Chapter 2, Ottaway and others [10] showed that all 16 possible combinations of outcomes are attainable in general: for example, there exist games in  $\mathcal{L}^- \cap \mathcal{L}^+$ . However, more than half of these pairs do not occur among ends. Since either Left or Right has no first move in an end  $G$ , that player wins immediately under misère rules and loses immediately under normal rules. Thus, an end cannot be a previous-win under misère play, nor a next-win under normal play, and furthermore an end cannot have the same outcome under both types of play. This fact, which is not exclusive to alternating games, is summarized in Lemma 3.2.1.

**Lemma 3.2.1.** *If  $G$  is an end then  $o^-(G) \neq \mathcal{P}^-$ ,  $o^+(G) \neq \mathcal{N}^+$ , and  $o^-(G) \neq o^+(G)$ .*

Note that Lemma 3.2.1 implies that if Left has a good (misère) first move in an alternating position, then the move is to a position in  $\mathcal{L}^-$ . This is because any left option of an alternating game is a left end, and no end can be in  $\mathcal{P}^-$ . Likewise, if Right has a good first move in an alternating game, it is to a position in  $\mathcal{R}^-$ . These properties of alternating games will be used repeatedly without reference.

With the majority of outcome pairs excluded by Lemma 3.2.1, seven possibilities for ends remain:  $\mathcal{R}^- \cap \mathcal{L}^+$ ,  $\mathcal{N}^- \cap \mathcal{L}^+$ ,  $\mathcal{R}^- \cap \mathcal{P}^+$ ,  $\mathcal{N}^- \cap \mathcal{P}^+$ ,  $\mathcal{L}^- \cap \mathcal{R}^+$ ,  $\mathcal{N}^- \cap \mathcal{R}^+$ , and  $\mathcal{L}^- \cap \mathcal{P}^+$ . Each of these occur in the universe  $\mathcal{A}_e$ , as illustrated in Figure 3.2. The representative positions in Figure 3.2 will be referred to frequently throughout this

chapter; they are the zero game and the games

$$A = \{0 \mid \cdot\}, B = \{0, \bar{A} \mid \cdot\}, C = \{\bar{B} \mid \cdot\},$$

along with the corresponding left-end conjugates

$$\bar{A} = \{\cdot \mid 0\}, \bar{B} = \{\cdot \mid 0, A\}, \bar{C} = \{\cdot \mid B\}.$$

Note that the game  $A$  is in fact the normal-play integer  $\mathbf{1}$ . In Chapter 5 there are multiple results about (normal-play) canonical-form integers, and so it is natural and convenient to use the normal-play names; in  $\mathcal{A}$ ,  $\{0 \mid \cdot\}$  is the only such position discussed, and so it is labelled ‘ $A$ ’ to be consistent with the other two left alternating ends ‘ $B$ ’ and ‘ $C$ ’.

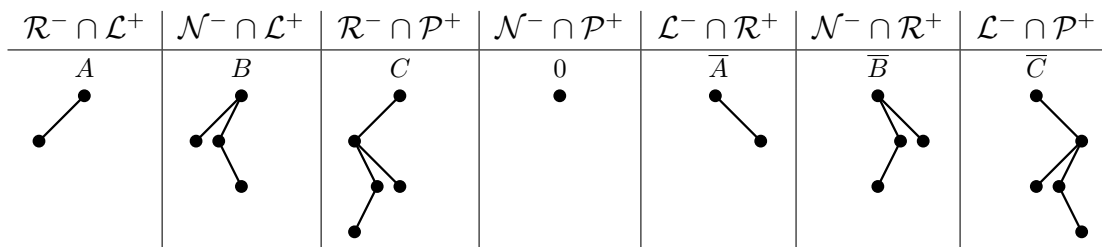


Figure 3.2: The possible outcome pairs for alternating ends.

Returning to our earlier example, note that most of the positions in Figure 3.2 appear in PENNY NIM: if we consider stacks on which Left can play, then a one-penny stack is the game  $A$  and a two-penny stack is the game  $B$ . In Section 3.5 we see that any other non-zero stack is equivalent (modulo  $\mathcal{A}$ ) to  $B$ .

The first major result of this section (Theorem 3.2.6) is that every alternating end is equivalent (modulo  $\mathcal{A}$ ) to all other alternating ends with the same pair of misère and normal outcomes. In particular, the seven games in Figure 3.2 are equivalent to all others in their respective classes. We first prove separately that every alternating end in  $\mathcal{N}^- \cap \mathcal{P}^+$  is equivalent to zero. To see both this and the remaining equivalencies, we must establish some domination of options among alternating ends.

Lemma 3.2.2 is required before the first of the domination results. It says that a sum of alternating positions that are all left ends is a win for Left as long as at least one of the individual positions is left-win. Intuitively, this is reasonable; at every turn, Left’s only option is to respond locally wherever Right has played (since

all other components are left ends), and eventually Right will play out the left-win position, to which Left has no response.

**Lemma 3.2.2.** *If  $X$  is a sum of alternating left ends with at least one component in  $\mathcal{L}^-$ , then  $X \in \mathcal{L}^-$ .*

*Proof.* Left wins trivially playing first on  $X$ . Assume, for followers of  $X$ , Left wins playing second on a sum of left ends when at least one of them is in  $\mathcal{L}^-$ . Let  $G$  be a component of  $X$  in  $\mathcal{L}^-$ . If Right plays in  $G$  to some  $G^R \in \mathcal{L}^- \cup \mathcal{N}^-$  then Left either has no response and wins immediately, or responds with a good move  $G^{RL} \in \mathcal{L}^-$  and wins by induction on the sum. If Right plays in some other component of  $X$  then Left either has no response and wins, or can play in that component to bring it back to a left end, and then wins playing second on the sum, by induction, since  $G$  is still in  $\mathcal{L}^-$ .  $\square$

Lemma 3.2.3 claims that in the alternating universe, a left end with normal outcome  $\mathcal{P}^+$  is at least as good for Left as the zero game. Left playing first trivially wins each game, and playing second cannot do worse on the nonzero game. If Left can win on a sum then Left can win on the sum plus a left end  $G \in \mathcal{P}^+$  by responding to any Right move in  $G$  with a winning normal-play strategy, thereby getting the last move in  $G$  and forcing Right to resume losing on the sum.

**Lemma 3.2.3.** *If  $G$  is an alternating left end in  $\mathcal{P}^+$  then  $G \geq 0 \pmod{\mathcal{A}}$ , and if  $G$  is an alternating right end in  $\mathcal{P}^+$  then  $G \leq 0 \pmod{\mathcal{A}}$ .*

*Proof.* We prove the first statement, and the second will follow by symmetry. Let  $G$  be an alternating left end with normal outcome  $\mathcal{P}^+$  and let  $X$  be a sum of alternating games. We need to show that  $o^-(G + X) \geq o^-(X)$ . If  $o^-(X) = R^-$  then trivially  $o^-(G + X) \geq o^-(X)$ . It remains to show that if Left can win  $X$  going first (second) then she can win  $G + X$  going first (second). Assume inductively that these statements hold for all followers of  $G + X$  of the form  $G' + X'$  with  $G' \in \mathcal{P}^+$ .

Suppose now that Left can win  $X$  going first. If she has no move in  $X$  then she has no move in  $G + X$  and so wins immediately. Otherwise, she has a move to some  $X^L$  from which she wins moving second, and by induction she then wins  $G + X^L$  moving second. Thus Left can win  $G + X$  moving first.

Suppose Left wins  $X$  moving second. Note that Right always has a move at the outset. If Right moves in  $G$  then Left has a (normal-play) winning local response to some  $G^{RL}$ , which as a left end must be in  $\mathcal{P}^+$ , not  $\mathcal{L}^+$ . Then Left wins  $G^{RL} + X$  by induction. If Right moves in  $X$  to  $X^R$  then either Left wins because she has no move, or she responds with her good second move  $X^{RL}$  and by induction wins  $G + X^{RL}$  moving second.  $\square$

We see from Lemma 3.2.3 that Left prefers (over zero) any left end that is previous-win under normal play. Now we will see that when an end has misère outcome  $\mathcal{N}^-$  in addition to normal outcome  $\mathcal{P}^+$ , *Right* actually prefers that end over zero, too. Together this shows that every alternating end in  $\mathcal{N}^- \cap \mathcal{P}^+$  is indistinguishable from zero, among *all* alternating games.

**Theorem 3.2.4.** *If  $G$  is any alternating end in  $\mathcal{N}^- \cap \mathcal{P}^+$ , then  $G \equiv 0 \pmod{\mathcal{A}}$ .*

*Proof.* Assume  $G$  is a right end in  $\mathcal{N}^- \cap \mathcal{P}^+$  (the other case is symmetric). By Lemma 3.2.3 we already have  $G \leq 0 \pmod{\mathcal{A}}$ , and so it suffices to show  $G \geq 0 \pmod{\mathcal{A}}$ . Let  $X$  be any sum of alternating games. Assume that for all followers  $X'$  of  $X$ , Left can win  $G + X'$  going first (second) when she can win  $X'$  going first (second).

Suppose Left wins  $X$  playing first. If she has no move in  $X$ , then in  $G + X$  she can make a good first misère move in  $G \in \mathcal{N}^-$ , to say  $G^L \in \mathcal{L}^-$ , to bring the whole position to a sum of left ends, one of which is in  $\mathcal{L}^-$ . Lemma 3.2.2 then shows Left wins playing first on  $G + X$ . Otherwise, Left has a move in  $X$  to some  $X^L$  from which she wins playing second; Left then wins  $G + X^L$  playing second, by induction, and so wins  $G + X$  playing first.

Suppose Left wins  $X$  playing second. Then Left can win  $X^R$  playing first for any Right move  $X^R$ , and so by induction wins  $G + X^R$  playing first. Since Right has no choice but to play in  $X$  on the sum  $G + X$ , this shows Left wins playing second on  $G + X$ .  $\square$

We want to show that every alternating end *not* in  $\mathcal{N}^- \cap \mathcal{P}^+$  is also equivalent to every other alternating end with the same pair of outcomes. We first establish two instances of domination among left options: Left should move to a position in  $\mathcal{L}^- \cap \mathcal{P}^+$  over one in  $\mathcal{N}^- \cap \mathcal{P}^+$ , and should choose  $\mathcal{N}^- \cap \mathcal{P}^+$  over a position in  $\mathcal{N}^- \cap \mathcal{R}^+$ .

Naively, Left prefers one option over another if the misère or normal outcome is more favourable for Left and the other outcome is just the same.

**Theorem 3.2.5.** *If  $G \in \mathcal{N}^- \cap \mathcal{R}^+$  and  $H \in \mathcal{L}^- \cap \mathcal{P}^+$  are alternating ends, then  $G \leq 0 \leq H \pmod{\mathcal{A}}$ .*

*Proof.* Note that  $G$  and  $H$  are necessarily left ends. Since  $H \in \mathcal{P}^+$ , Lemma 3.2.2 gives us  $0 \leq H$ . For  $G \leq 0$ , let  $X$  be a sum in  $\mathcal{A}$ . It suffices to show that if Right can win  $X$  playing first (second) then Right can win  $G + X$  playing first (second); assume this is true for all followers  $X'$  of  $X$ . Now suppose Right can win  $X$  playing first. If Right has no moves in  $X$ , he can make a good first misère move in  $G$  to  $G^R \in \mathcal{R}^-$  and win on  $G^R + X$  by Lemma 3.2.2. Otherwise, Right makes his good misère move in  $X$  to  $X^R$ ; since Right wins  $X^R$  playing second, he wins  $G + X^R$  playing second by induction. This shows Right wins  $G + X$  playing first.

If Right wins  $X$  playing second then Left has no good first move in  $X$  and cannot play in the left end  $G$ . Left must move  $X$  to  $X^L$ , which Right can win playing first, and then by induction Right can win  $G + X^L$  playing first. Thus Right wins  $G + X$  playing second.  $\square$

The symmetric result obviously holds as well, and together these give us a ‘partial’ partial order of moves, as illustrated in Figure 3.3. In particular, Theorem 3.2.5 tells us that  $C \leq 0 \leq B$  and  $\bar{B} \leq 0 \leq \bar{C}$ . We will be able to establish another chain of domination ( $C \leq A \leq B$ ,  $\bar{B} \leq \bar{A} \leq \bar{C}$ ) after we prove Theorem 3.2.6.

Theorem 3.2.6(i) uses the limited misère and normal outcomes of  $\mathcal{A}_e$  (Lemma 3.2.1) to argue that any end in  $\mathcal{R}^- \cap \mathcal{L}^+$  has a left option to zero, which, by Theorem 3.2.5, dominates all other potential options; this shows the end is equivalent to  $\{0 \mid \cdot\} = A$ . Statement (ii) is symmetric. The remaining four cases follow similar arguments, but rely upon induction in order to reduce each left or right option to one of  $A, B, C$ , or their conjugates.

**Theorem 3.2.6.** *If  $G$  is an alternating end, then the following statements are true modulo  $\mathcal{A}$ .*

- |                                                                       |                                                                            |
|-----------------------------------------------------------------------|----------------------------------------------------------------------------|
| (i) If $G \in \mathcal{R}^- \cap \mathcal{L}^+$ then $G \equiv A$ .   | (ii) If $G \in \mathcal{L}^- \cap \mathcal{R}^+$ then $G \equiv \bar{A}$ . |
| (iii) If $G \in \mathcal{N}^- \cap \mathcal{L}^+$ then $G \equiv B$ . | (iv) If $G \in \mathcal{N}^- \cap \mathcal{R}^+$ then $G \equiv \bar{B}$ . |
| (v) If $G \in \mathcal{R}^- \cap \mathcal{P}^+$ then $G \equiv C$ .   | (vi) If $G \in \mathcal{L}^- \cap \mathcal{P}^+$ then $G \equiv \bar{C}$ . |

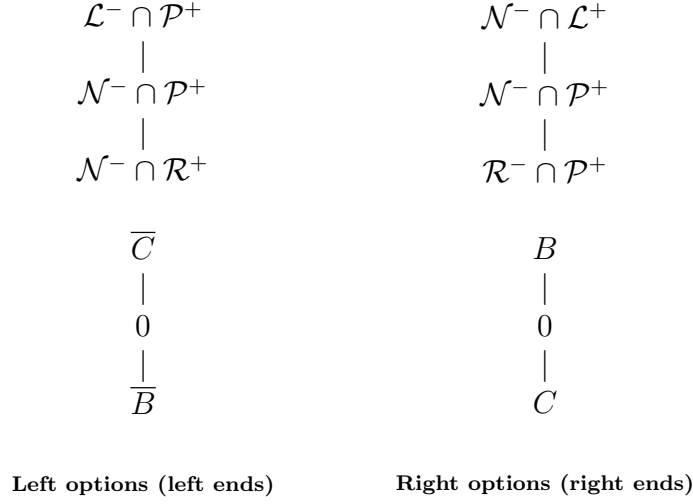


Figure 3.3: Dominated left and right options of alternating games.

The base cases for these inductions are shown separately in Lemma 3.2.7. By symmetry we need only demonstrate that (iii) and (v) hold.

**Lemma 3.2.7.** *If  $G \in \mathcal{N}^- \cap \mathcal{L}^+$  is an alternating end born by day 3 then  $G \equiv B \pmod{\mathcal{A}}$ , and if  $G \in \mathcal{R}^- \cap \mathcal{P}^+$  is an alternating end born by day 3 then  $G \equiv C \pmod{\mathcal{A}}$ .*

*Proof.* If  $G$  is in either of  $\mathcal{N}^- \cap \mathcal{L}^+$  or  $\mathcal{R}^- \cap \mathcal{P}^+$  then  $G$  is a right end. If  $G$  is alternating, and born by day 3, then the possible left options  $G^L$  are  $0$ ,  $\overline{A}$ ,  $\overline{B}$ , or  $\{\cdot \mid A\}$ . The last of these is actually equivalent to zero by Theorem 3.2.4, because its outcome pair is  $\mathcal{N}^- \cap \mathcal{P}^+$ . The possible sets of left options  $G^L$  are thus  $\{0\}$ ,  $\{\overline{A}\}$ ,  $\{\overline{B}\}$ ,  $\{0, \overline{A}\}$ ,  $\{0, \overline{B}\}$ ,  $\{\overline{A}, \overline{B}\}$ , and  $\{0, \overline{A}, B\}$ . Using the domination of  $0$  over  $\overline{B}$ , we can reduce these possibilities to  $\{0\}$ ,  $\{\overline{A}\}$ ,  $\{\overline{B}\}$ ,  $\{0, \overline{A}\}$ , and  $\{\overline{A}, \overline{B}\}$ . This means  $G$  is one of

$$\begin{aligned}
 \{0 \mid \cdot\} &= A \in \mathcal{R}^- \cap \mathcal{L}^+, \\
 \{\overline{A} \mid \cdot\} &\in \mathcal{N}^- \cap \mathcal{P}^+, \\
 \{\overline{B} \mid \cdot\} &= C \in \mathcal{R}^- \cap \mathcal{P}^+, \\
 \{0, \overline{A} \mid \cdot\} &= B \in \mathcal{N}^- \cap \mathcal{L}^+, \\
 \{\overline{A}, \overline{B} \mid \cdot\} &\in \mathcal{N}^- \cap \mathcal{P}^+,
 \end{aligned}$$

and so if  $G$  is born by day 3 then  $G \in \mathcal{N}^- \cap \mathcal{L}^+$  implies  $G \equiv B$ , and  $G \in \mathcal{R}^- \cap \mathcal{P}^+$  implies  $G \equiv C$ .  $\square$

*Proof of Theorem 3.2.6.*

(i)–(ii): If  $G \in \mathcal{R}^- \cap \mathcal{L}^+$  then  $G$  is a right end, and since Left has no good first misère move, every Left option must be in  $\mathcal{N}^-$  or  $\mathcal{R}^-$ . But all Left options are left ends (or zero), and thus cannot be in  $\mathcal{R}^-$ . So every Left option is in  $\mathcal{N}^-$ . Similarly, since  $G \in \mathcal{L}^+$  there is at least one Left option in  $\mathcal{P}^+$  or  $\mathcal{L}^+$ ; but left ends (or zero) cannot be in  $\mathcal{L}^+$ , so at least one Left option is in  $\mathcal{P}^+$ . Together this shows there exists a  $G^L \equiv 0 \in \mathcal{N}^- \cap \mathcal{P}^+$ . By Theorem 3.2.5, any other options (necessarily in  $\mathcal{N}^- \cap \mathcal{R}^+$ ) are dominated by 0, and so  $G \equiv \{0 \mid \cdot\} \equiv A$ . Case (ii) follows by symmetry.

(iii)–(vi): We combine the remaining results and use an inductive proof. Lemma 3.2.7 shows that the only alternating ends born by day 3 in  $\mathcal{N}^- \cap \mathcal{L}^+$  (respectively  $\mathcal{R}^- \cap \mathcal{P}^+$ ,  $\mathcal{N}^- \cap \mathcal{R}^+$ ,  $\mathcal{L}^- \cap \mathcal{P}^+$ ) are equivalent to  $B$  (respectively  $\overline{B}, C, \overline{C}$ ), so the base cases hold.

Assume statements (iii)–(vi) hold for all ends born by day  $n \in \mathbb{N}$ . Let  $G$  and  $H$  be games in  $\mathcal{R}^- \cap \mathcal{P}^+$  and  $\mathcal{N}^- \cap \mathcal{L}^+$ , respectively, born on day  $n + 1$  (note that  $G$  and  $H$  are right ends; the left end results follow similarly). As argued in (i)–(ii), every Left option of  $G$  must be in  $\mathcal{N}^-$  because Left has no good first misère move in  $G \in \mathcal{R}^-$ . Additionally,  $G \in \mathcal{P}^+$  forces every Left option to be in  $\mathcal{R}^+$  or  $\mathcal{N}^+$ ; but ends cannot be in  $\mathcal{N}^+$ , so every option is in  $\mathcal{N}^- \cap \mathcal{R}^+$ . By induction each of these options is indistinguishable from  $\overline{B}$ . Thus  $G \equiv \{\overline{B} \mid \cdot\} = C$ .

For  $H$  there is a bit more to show. Since  $H \in \mathcal{N}^-$ , Left has a good misère move to a left end in  $\mathcal{L}^-$ ; since  $H \in \mathcal{L}^+$ , Left has a good normal-play move to a left end in  $\mathcal{P}^+$ . This shows that Left either has a move to  $\mathcal{L}^- \cap \mathcal{P}^+$  ( $\equiv \overline{C}$ , by induction) or has moves to both  $\mathcal{L}^- \cap \mathcal{R}^+$  ( $\equiv \overline{A}$  by (i)) and  $\mathcal{N}^- \cap \mathcal{P}^+$  ( $\equiv 0$ ). If there are any moves to  $\mathcal{N}^- \cap \mathcal{R}^+$ , they are dominated by 0 or  $\overline{C}$ , by Theorem 3.2.5. Using this and the domination of  $\overline{C}$  over 0, we reduce the possibilities for  $H$  to

$$H \equiv \{\overline{A}, 0 \mid \cdot\}, \{\overline{C} \mid \cdot\}, \text{ or } \{\overline{A}, \overline{C} \mid \cdot\}.$$

In the first case we have  $H \equiv B$  immediately. It remains to show that  $\{\overline{C} \mid \cdot\} \equiv B$  and  $\{\overline{A}, \overline{C} \mid \cdot\} \equiv B$ .

*Claim 1:*  $\{\overline{C} \mid \cdot\} \equiv B$ .

Let  $J = \{\overline{C} \mid \cdot\}$ . We need to show  $o^-(B + X) = o^-(J + X)$  for all sums  $X$  in  $\mathcal{A}$ . Assume true for all followers of  $X$ . If Right wins playing first in  $B + X$  then he must win second from some  $B + X^R$ , which by induction has the same outcome as  $J + X^R$ , and then Right wins first from  $J + X$  with the same move in  $X$ . If Left wins playing second in  $B + X$  the same argument shows that Left wins playing second in  $J + X$ , since for every  $X^R$ ,  $o^-(B + X^R) = o^-(J + X^R)$ .

If Right wins playing second on  $B + X$ , and Left's first move in  $J + X$  is to some  $J + X^L$ , then since Right can win playing first on  $B + X^L$  he can win playing first from  $J + X^L$ , by induction. Otherwise, Left's first move in  $J + X$  is to  $\overline{C} + X$ , and Right should respond with  $B + X$ , from which he wins playing second.

Finally, suppose Left wins  $B + X$  playing first. If her good move is in  $X$  then as above Left wins  $J + X$  with the same first move. If Left's good first move is to  $B^L + X$  then Left must be able to win playing second from any position  $B^L + X^{RL\dots RL}$  obtained from optimal Left play (including  $B^L + X$ ). Now, in  $J + X$ , Left should move first to  $\overline{C} + X$  and play her original strategy in  $X$ . When Right chooses to play in  $\overline{C}$ , to  $B + X^{RL\dots RL}$  for some (not necessarily proper) follower of  $X$ , Left plays to  $B^L + X^{RL\dots RL}$  and wins from there playing second.

*Claim 2:*  $\{\overline{A}, \overline{C} \mid \cdot\} \equiv B$ .

Let  $K = \{\overline{A}, \overline{C} \mid \cdot\}$ . Recall that when at least one left option already exists, introducing another left option cannot create a position worse for Left; Left simply 'ties her hand' and ignores the extra option. In this way we see that  $K \geq J \equiv B$ . It remains to show that  $K \leq B$ ; i.e., that Right wins  $K + X$  whenever he wins  $B + X$ .

If Right wins  $B + X$  playing first then he wins  $K + X$  playing first, by induction, as in claim 1. Suppose Right wins  $B + X$  playing second. If Left moves first in  $K + X$  to  $\overline{C} + X$  then Right wins by moving  $\overline{C}$  to  $B$  and then playing second on  $B + X$ . If Left moves first in  $K + X$  to  $\overline{A} + X$  then Right can win first from that position, since  $\overline{A} + X$  is a possible first Left move from  $B + X$ .  $\square$

Theorem 3.2.6 is a powerful result. Since the equivalences hold modulo *all* alternating games, and since every alternating game becomes an end after the first move, these equivalences give significantly reduced options in the universe  $\mathcal{A}$ . As promised, Theorem 3.2.6 also allows us to prove the missing chain of domination; we can then

combine the results of Theorems 3.2.5 and 3.2.8 (below) to obtain the partial orders illustrated in Figure 3.4. As indicated by the figure, the game 0 is incomparable with both  $A$  and  $\bar{A}$ , modulo  $\mathcal{A}$ . To see this, note that  $0 + * \in \mathcal{P}^-$  while  $A + * \in \mathcal{N}^-$ .

**Theorem 3.2.8.** *If  $G \in \mathcal{N}^- \cap \mathcal{R}^+$ ,  $H \in \mathcal{L}^- \cap \mathcal{P}^+$ , and  $K \in \mathcal{L}^- \cap \mathcal{R}^+$  are alternating ends, then  $G \leq K \leq H$  modulo  $\mathcal{A}$ .*

*Proof.* By Theorem 3.2.6 we need only show  $\bar{B} \leq \bar{A} \leq \bar{C}$ . As in the proof of Claim 2 above, we immediately have  $\bar{B} \leq \bar{A}$ , since Right can ‘tie his hand’ in  $\bar{B} = \{\cdot \mid 0, A\}$  and pretend he is playing with the position  $\bar{A} = \{\cdot \mid 0\}$ ; that is,  $\bar{B}$  is at least as good as  $\bar{A}$  for Right.

To see that  $\bar{A} \leq \bar{C}$ , let  $X$  be any sum of alternating games, and suppose Left wins  $\bar{A} + X$  playing first. Then we know Left must be able to win playing second from  $\bar{A} + X^{LR\dots L}$ , for any  $X^{LR\dots L}$  obtained from  $X$  under optimal play. In  $\bar{C} + X$ , Left plays as usual until Right moves from  $\bar{C} + X^{LR\dots L}$  to  $B + X^{LR\dots L}$ ; Left then moves to  $\bar{A} + X^{LR\dots L}$  and wins playing second from there. A similar argument shows Left wins  $\bar{C} + X$  playing second whenever she wins  $\bar{A} + X$  playing second, and so  $\bar{A} \leq \bar{C}$ .  $\square$

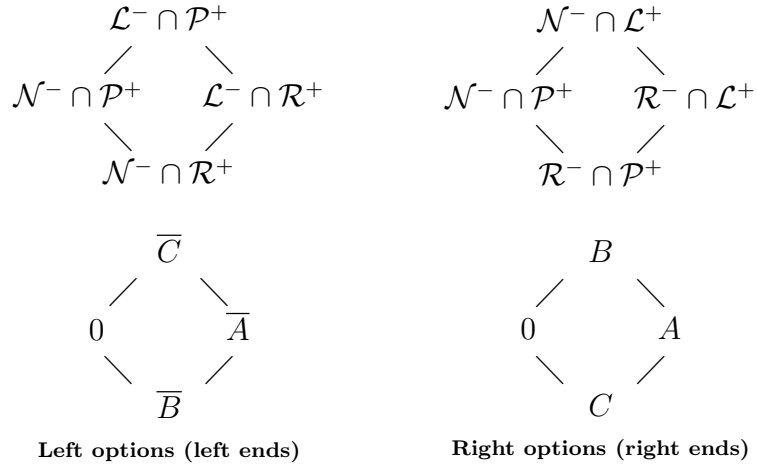


Figure 3.4: The partial orders (modulo  $\mathcal{A}$ ) given by Theorems 3.2.5 and 3.2.8.

We move now from individual alternating ends to disjunctive sums of these positions. Theorem 3.2.9 addresses one of the main difficulties in misère game theory: the lack of inverses under disjunctive sum. As mentioned in the Chapter 2, we do not generally have  $G + \bar{G} = 0$  in misère play. Fortunately, alternating ends do possess this very convenient property.

**Theorem 3.2.9.** *If  $G$  is an alternating end then  $G + \overline{G} \equiv 0 \pmod{\mathcal{A}}$ .*

*Proof.* Since  $\mathcal{A}$  is closed under conjugates and  $\mathcal{A}_e$  is closed under followers, we can apply Theorem 2.4.2. Let  $X$  be any left end in  $\mathcal{A}$  (thus in  $\mathcal{A}_e$ ). It suffices to show that  $G + \overline{G} + X$  has a winning first move for Left. If  $G \in \mathcal{N}^- \cap \mathcal{P}^+$  then we already know  $G \equiv 0$  and so  $G + \overline{G} + X \equiv X \in \mathcal{L}^- \cup \mathcal{N}^-$  (as it is a left end). If  $G$  is any other end, without loss of generality a right end, then  $G \in \mathcal{R}^-$  and the left end  $\overline{G}$  is in  $\mathcal{L}^-$ . Left can move in  $G$  to leave a sum of all left ends, one of which is in  $\mathcal{L}^-$ , and wins by Lemma 3.2.2. This proves that  $G + \overline{G} \equiv 0 \pmod{\mathcal{A}}$  for every  $G \in \mathcal{A}_e$ .  $\square$

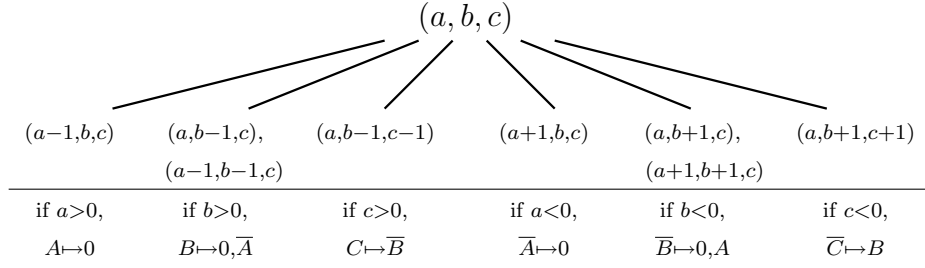
Theorem 3.2.9 shows that disjunctive sum makes the set of alternating ends into a group. Since  $\overline{G}$  is the additive inverse of  $G$ , we can safely write  $\overline{G} = -G$ . We use the notations interchangeably for the remainder of the chapter and write  $-kG$  to represent  $k$  copies of the game  $-G$ .

Together, Theorems 3.2.6 and 3.2.9 show that any sum of ends in the alternating universe  $\mathcal{A}$  can be written as  $aA + bB + cC$  for integers  $a, b, c$ . We can therefore represent such a sum as an ordered triple  $(a, b, c)$ . What are the left and right options from this position?

- If  $a > 0$ , then Left can move an  $A$  position to 0, which leaves the triple  $(a - 1, b, c)$ .
- If  $b > 0$ , then Left can move a  $B$  position to 0, leaving  $(a, b - 1, c)$ , or to  $\overline{A} = -A$ , leaving  $(a - 1, b - 1, c)$ .
- If  $c > 0$ , then Left can move a  $C$  position to  $\overline{B} = -B$ , leaving  $(a, b - 1, c - 1)$ .

The options for Right are obviously symmetric. Figure 3.5 illustrates these options from  $(a, b, c)$  (with the conditions for each indicated below the horizontal line).

Notice that if Left has a move (that is, if at least one of  $a, b, c$  is positive), then Left has a move that reduces either  $a$  or  $c$  by 1, and thus Left can always reduce the sum  $a + c$  by 1. Similarly, if Right has a move then Right can increase  $a + c$  by 1. Suppose the players rally back and forth in this way and Left runs out of moves first. Right's previous turn must have ended with  $a + c \leq 0$  (else Left still has a move); but since  $a + c$  has been alternating by  $\pm 1$ , this means Right ended every turn with  $a + c \leq 0$ , or equivalently began every turn with  $a + c < 0$ . We might guess that a

Figure 3.5: Left and right options from  $aA + bB + cC$ .

position with  $a + c < 0$  at the outset is a win for Left; in fact this is precisely the case, as stated below in Theorem 3.2.10.

**Theorem 3.2.10.** *Let  $G = aA + bB + cC$ . Then*

$$o^-(G) = \begin{cases} \mathcal{L}^-, & \text{if } a + c < 0, \\ \mathcal{N}^-, & \text{if } a + c = 0, \\ \mathcal{R}^-, & \text{if } a + c > 0. \end{cases}$$

*Proof.* The zero game is a next-player win under misère rules and has  $a + c = 0$ . Assume the outcomes above hold for all followers of  $G = (a, b, c)$ .

If  $a + c < 0$  then there exists at least one copy of  $\bar{A}$  or  $\bar{C}$  in the sum, so Right has a move going first to  $(a + 1, b, c)$  or  $(a, b + 1, c + 1)$ . At best then, Right can increase the value of  $a + c$  to zero, giving Left a next-win position by induction. If Left plays first she can play to either  $(a - 1, b, c)$  or  $(a, b - 1, c - 1)$ , and so can guarantee  $a + c$  remains negative and by induction leaves a Left-win position. Since Left wins playing first or second,  $G \in \mathcal{L}^-$  when  $a + c < 0$ . A symmetric argument shows  $G \in \mathcal{R}^-$  when  $a + c > 0$ .

If  $a + c = 0$  then Left going first either has no moves, and wins immediately, or has a move in a copy of  $A, B$ , or  $C$ . From Figure 3.5 we see Left can always decrease either  $a$  or  $c$  by 1, thereby moving to  $a + c < 0$  and winning by induction. Right wins playing first by symmetry, and so  $G \in \mathcal{N}^-$  when  $a + c = 0$ .  $\square$

Theorem 3.2.10 shows that the equivalence classes among alternating ends collapse even further when the universe is restricted to  $\mathcal{A}_e$ . Notice that the integer  $b$  does not influence the outcome of  $G = aA + bB + cC$ . Thus, for  $X$  a sum of alternating ends, we have  $o^-(X) = o^-(B + X)$ . This gives the following corollaries.

**Corollary 3.2.11.** *If  $\mathcal{A}_e$  is the universe of alternating ends, then  $B \equiv 0 \pmod{\mathcal{A}_e}$  and  $C \equiv A \pmod{\mathcal{A}_e}$ .*

**Corollary 3.2.12.** *If  $G$  is a sum of alternating ends then  $G \equiv aA \pmod{\mathcal{A}_e}$  for some integer  $a$ .*

Note that we do have to restrict the universe to  $\mathcal{A}_e$  to get these results. In the larger universe  $\mathcal{A}$ ,  $B$  is distinguishable from 0 with the game  $* = \{0 \mid 0\}$ : Left can win  $B + *$  going first but loses playing first on  $*$ .

The last thing to note in the analysis of  $\mathcal{A}_e$  is that the positions  $aA$  and  $a'A$  are equivalent if and only if  $a = a'$ . If  $a \neq a'$  then the games are distinguished by  $-aA$ , since  $aA + (-aA)$  is in  $\mathcal{N}^-$  while  $a'A + (-aA)$  is in  $\mathcal{L}^-$  or  $\mathcal{R}^-$ .

We can now describe the monoid of the alternating end universe  $\mathcal{A}_e$ .

$$\mathcal{M}_{\mathcal{A}_e} = \langle 0, A, \bar{A} \mid A + \bar{A} = 0 \rangle,$$

$$\mathcal{N}^- = \{0\}, \mathcal{P}^- = \emptyset, \mathcal{R}^- = \{aA \mid a \in \mathbb{N}\}, \mathcal{L}^- = \{(-aA) \mid a \in \mathbb{N}\}.$$

As noted earlier, this monoid is actually a group; the mapping  $aA \mapsto a$  for  $a \in \mathbb{Z}$  shows  $\mathcal{M}_{\mathcal{A}_e} \cong (\mathbb{Z}, +)$ .

### 3.3 Equivalence Classes of Alternating Games

In this section we determine the equivalence classes of individual alternating games. The results of the previous section, particularly Theorem 3.2.6, are very useful here. In Section 3.4 this work is continued by considering the outcomes of *sums* of alternating games.

Every option of an alternating game is an alternating end. Recall that Theorem 3.2.6, which gives the equivalence classes of individual alternating ends, actually holds modulo  $\mathcal{A}$ ; thus the only possible left options of an alternating game are  $0, \bar{1}, \bar{B}$ , or  $\bar{C}$ , and the only possible right options are the conjugates of these games.

We now use the domination results of Theorem 3.2.5 and 3.2.8 to further reduce the number of distinguishable alternating games. Since  $\bar{C}$  dominates all other left options, the only positions with a left option to  $\bar{C}$  are those with  $G^{\mathcal{L}} = \{\bar{C}\}$ . If Left has no option to  $\bar{C}$  then the possibilities for  $G^{\mathcal{L}}$  are listed below in Lemma 3.3.1, with the equivalence (modulo  $\mathcal{A}$ ) following from Theorem 3.2.5. Since we have already

established the equivalences of ends in  $\mathcal{A}$ , assume  $G$  is not an end (that is, assume neither  $G^L$  nor  $G^R$  is empty).

**Lemma 3.3.1.** *If  $G$  is a non-end alternating game then the possible sets of left options  $G^L$  are*

1.  $\{\overline{C}\}$
2.  $\{0, \overline{A}, \overline{B}\} \equiv \{0, \overline{A}\}$
3.  $\{\overline{A}, \overline{B}\} \equiv \{\overline{A}\}$
4.  $\{0, \overline{B}\} \equiv \{0\}$
5.  $\{\overline{B}\}$

Thus, there are five distinct sets of options for Left:  $\{\overline{C}\}$ ,  $\{0, \overline{A}\}$ ,  $\{\overline{A}\}$ ,  $\{0\}$ , and  $\{\overline{B}\}$ . Similarly, there are five possibilities for the set of right options  $G^R$ . The resulting 25 possible non-end alternating positions are shown in Figure 3.6.

Note immediately that the position  $\{\overline{A} \mid A\}$  has the same game tree as  $A + \overline{A} \equiv 0$  (by Theorem 3.2.9). This fact strengthens a result from the previous section, giving Lemma 3.3.2 below. We will see shortly that  $G \in \mathcal{N}^- \cap \mathcal{P}^+$  is in fact both necessary and sufficient for  $G \equiv 0 \pmod{\mathcal{A}}$ .

**Lemma 3.3.2.** *If  $G$  is any alternating position  $\mathcal{N}^- \cap \mathcal{P}^+$  then  $G \equiv 0 \pmod{\mathcal{A}}$ .*

There is another equivalency which is not so immediate as  $\{\overline{A} \mid A\} \equiv 0$ : the position  $\{\overline{A} \mid B\}$  is indistinguishable from the alternating end  $\overline{C}$ . This equivalency is not entirely surprising, as we have seen that a left option to  $\overline{A}$  alone is often irrelevant; for example,  $\{\overline{A} \mid \cdot\} \equiv \{\cdot \mid \cdot\} \equiv \{\overline{A} \mid A\}$ . However, the result is more subtle than this, as it is *not* the case that  $\{\overline{A} \mid 0, A\}$  is equivalent to  $B = \{\cdot \mid 0, A\}$ , nor is  $\{\overline{A} \mid C\}$  equivalent to  $\{\cdot \mid C\} \equiv \overline{B}$ . Both pairs are distinguished shortly. The difference, for  $\{\overline{A} \mid B\}$  and  $\overline{C} = \{\cdot \mid B\}$ , is that the right option of these games is not good for Right; in particular, Left can respond to zero if Right moves in  $\overline{C}$ . The advantage of this move can be seen in the proof of Lemma 3.3.3 below.

**Lemma 3.3.3.**  *$\{\overline{B} \mid A\} \equiv C$  and  $\{\overline{A} \mid B\} \equiv \overline{C} \pmod{\mathcal{A}}$ .*

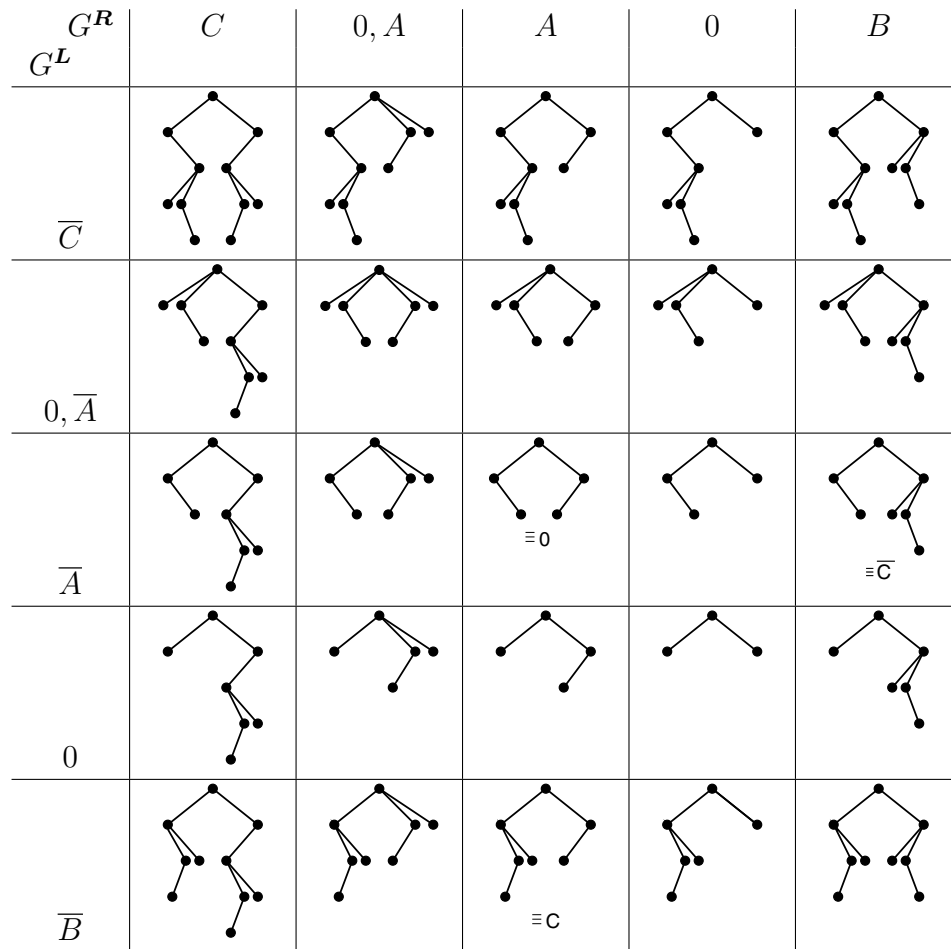


Figure 3.6: All possible non-end alternating positions.

*Proof.* Let  $G = \{\bar{A} \mid B\}$ , and let  $X$  be any sum of alternating games. Suppose Left wins  $\bar{C} + X$ . To win  $G + X$ , if  $X$  is not already a left end, Left plays her winning strategy from  $\bar{C} + X$ . If Right plays in  $G$ , then the resulting position is a possible follower of  $\bar{C} + X$ , and so Left can win from there. Otherwise,  $X$  is eventually played to a sum of left ends, with Left to move. Left then plays in  $G$  to  $\bar{A}$ , leaving a sum of left ends with one in  $\mathcal{L}^-$ , which is a left-win position.

Suppose now that Left wins  $G + X$ ; again, Left plays the same strategy in  $\bar{C} + X$ . If at some point, say  $G + X'$ , that strategy would have Left play in  $G$  to  $\bar{A} + X'$ , then  $0 + X'$  is a possible Right response, meaning Left must be able to win  $X'$  playing first. It remains to show that  $\bar{C} + X'$  is in  $\mathcal{L}^- \cup \mathcal{N}^-$  when  $X'$  is. To see this, note that Left can play as usual in  $X'$ , until Right plays in  $\bar{C}$  to  $B$ , at which point Left plays  $B$  to zero and forces play to resume in  $X'$ , where she wins as the second player.  $\square$

After reducing  $\{\bar{A} \mid A\}$  to zero and eliminating the two non-ends of Lemma 3.3.3, we see there are at most 22 distinct non-end alternating positions. Thus, including the seven disjoint equivalence classes of alternating ends in  $\mathcal{A}$ , there are at most 29 disjoint equivalence classes among *individual* alternating positions. Representatives of these potential classes are shown in Figure 3.7 (note that the table is colour-coded to indicate the misère outcome of each position, as illustrated in Figure 3.8). In fact, there are *exactly* 29 disjoint equivalence classes: the positions in Figure 3.7 are pairwise distinguishable. This is stated as Theorem 3.3.4 below, which the remainder of the section serves to prove.

**Theorem 3.3.4.** *There are 29 disjoint equivalence classes of individual alternating games in the universe  $\mathcal{A}$ .*

Note that these are not all of the equivalence classes of the entire universe  $\mathcal{A}$ , which is the *closure* of these individual alternating games. Indeed, there are infinitely many classes in  $\mathcal{A}$ , since  $\mathcal{A}_e \cong \mathbb{Z}$  is a subset.

We now set about demonstrating the pairwise distinguishability of the 29 alternating positions. First note that any two positions with different misère outcomes are immediately distinguishable (by zero). Thus, we need only show that each group of positions with the same outcome are pairwise distinguishable. Let us begin with the group of next-win positions, which are shown again in Figure 3.9, along with their

$G^R$	$C$	$0, A$	$A$	$\cdot$	$0$	$B$
$\bar{C}$						
$0, \bar{A}$						
$\bar{A}$						
$\cdot$						
$0$						
$\bar{B}$						

Figure 3.7: The 29 distinct alternating positions.

$G^R$	$C$	$0, A$	$A$	$\cdot$	$0$	$B$
$\bar{C}$						
$0, \bar{A}$		$\mathcal{N}^-$			$\mathcal{L}^-$	
$\bar{A}$						
$\cdot$						
$0$		$\mathcal{R}^-$			$\mathcal{P}^-$	
$\bar{B}$						

Figure 3.8: Misère outcomes of the alternating positions of Figure 3.7.

misère and normal outcomes. When two next-win positions have different *normal* outcomes, they are distinguished by the alternating game  $*$ ; this fact, which is Lemma 3.3.5, demonstrates the distinguishability of 28 of the 36 pairs of next-win games. In particular, it shows that no other alternating positions, besides those accounted for in Lemma 3.3.2, are equivalent to zero modulo  $\mathcal{A}$ .

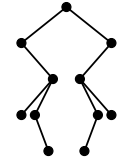
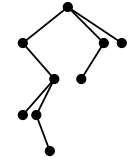
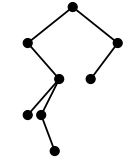
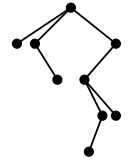
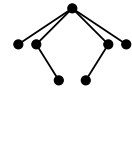
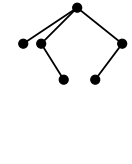
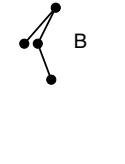
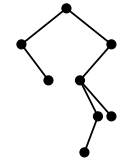
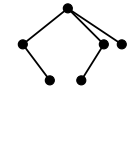
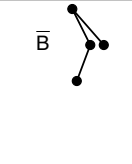
$G^R$	$C$	$0, A$	$A$	$\cdot$
$\bar{C}$	 $\mathcal{N}^- \cap \mathcal{N}^+$	 $\mathcal{N}^- \cap \mathcal{N}^+$	 $\mathcal{N}^- \cap \mathcal{L}^+$	
$0, \bar{A}$	 $\mathcal{N}^- \cap \mathcal{N}^+$	 $\mathcal{N}^- \cap \mathcal{N}^+$	 $\mathcal{N}^- \cap \mathcal{L}^+$	 $B$ $\mathcal{N}^- \cap \mathcal{L}^+$
$\bar{A}$	 $\mathcal{N}^- \cap \mathcal{R}^+$	 $\mathcal{N}^- \cap \mathcal{R}^+$		
$\cdot$		 $\bar{B}$ $\mathcal{N}^- \cap \mathcal{R}^+$		$\bullet$ $0$ $\mathcal{N}^- \cap \mathcal{P}^+$

Figure 3.9: Next-win alternating games and their normal-play outcomes.

**Lemma 3.3.5.** *If  $G \in \mathcal{N}^-$  is alternating position then  $o^-(G + *) = o^+(G)$ . In particular, if  $G, H \in \mathcal{N}^-$  with  $o^+(G) \neq o^+(H)$ , then  $G$  and  $H$  are distinguished by  $*$ .*

*Proof.* Neither player wins  $G + *$  by first moving in  $*$ , since  $G \in \mathcal{N}^-$ . Once the first move is made in  $G$ , if either player moves  $*$  to 0 then the other player wins from the lack of a consecutive move in  $G$ . Thus the winner will be the player who can take the last move in  $G$ , or, equivalently, the player who wins  $G$  under normal play. If

$G, H \in \mathcal{N}^-$  have different normal-play outcomes then  $o^-(G + *) = o^+(G) \neq o^+(H) = o^-(H + *)$ , so  $G \not\equiv H \pmod{\mathcal{A}}$ .  $\square$

**Corollary 3.3.6.**  $G \equiv 0 \pmod{\mathcal{A}}$  if and only if  $G \in \mathcal{N}^- \cap \mathcal{P}^+$ .

Lemma 3.3.5 proves the distinguishability of all but twelve pairs of next-win games. It remains to consider those pairs with the same normal-play outcome. There are six pairs of games in  $\mathcal{N}^- \cap \mathcal{N}^+$  and three in each of  $\mathcal{N}^- \cap \mathcal{R}^+$  and  $\mathcal{N}^- \cap \mathcal{L}^+$ . The distinguishability of seven of these pairs is demonstrated below (the others follow by symmetry); in each case the distinguishing game or sum given is the simplest possible<sup>2</sup>.

- $\{\bar{C} \mid C\} + \bar{C} + * \in \mathcal{N}^-$
  - $\{\bar{C} \mid 0, A\} + \bar{C} + * \in \mathcal{L}^-$
- $\{\bar{C} \mid A\} + C + * \in \mathcal{N}^-$
  - $\{0, \bar{A} \mid A\} + C + * \in \mathcal{R}^-$
- $\{\bar{C} \mid C\} + C + * \in \mathcal{N}^-$
  - $\{0, \bar{A} \mid 0, A\} + C + * \in \mathcal{R}^-$
- $\{\bar{C} \mid A\} + \bar{A} + * \in \mathcal{N}^-$
  - $B + \bar{A} + * \in \mathcal{L}^-$
- $\{\bar{C} \mid 0, A\} + C + * \in \mathcal{N}^-$
  - $\{0, \bar{A} \mid C\} + C + * \in \mathcal{R}^-$
- $\{0, \bar{A} \mid A\} + \bar{A} + * \in \mathcal{N}^-$
  - $B + \bar{A} + * \in \mathcal{L}^-$
- $\{\bar{C} \mid 0, A\} + C + * \in \mathcal{N}^-$
  - $\{0, \bar{A} \mid 0, A\} + C + * \in \mathcal{R}^-$

Next we compare pairs of alternating positions in  $\mathcal{L}^-$ . Our first observation is that left-win alternating games with different sets of right options are distinguished by  $*$ .

**Lemma 3.3.7.** If  $G, H \in \mathcal{L}^-$  and  $G^{\mathbf{R}} \neq H^{\mathbf{R}}$  then  $G$  and  $H$  are distinguished by  $*$ .

*Proof.* Without loss of generality, assume  $G^{\mathbf{R}} = \{0\}$  and  $H^{\mathbf{R}} = \{B\}$ . Right can win  $G + *$  playing first by moving to  $0 + * \in \mathcal{P}^-$ . However, Right has no good first move in  $H + *$ : moving in  $*$  leaves  $H \in \mathcal{L}^-$  and moving in  $H$  leaves  $B + *$ , which Left can bring to  $*$ .  $\square$

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<sup>2</sup>A computer program was written to identify the simplest distinguishing game for each pair of positions.



to establish a few basic results, which in turn are used in Section 3.5 to analyze the subuniverse of PENNY NIM positions.

### 3.4 Sums of Alternating Games

The main purpose of this section is to determine the outcome of a sum of a single alternating position with any number of alternating ends. As a corollary we get the invertibility of almost all alternating positions. These results provide a complete solution to the game PENNY NIM, which is presented in Section 3.5.

To proceed, we need a new piece of terminology and notation. As we saw in Section 3.2, the outcome of a sum of ends  $aA + bB + cC$  in  $\mathcal{A}$  is determined by the value  $a + c$ . When playing  $G + aA + bB + cC$ , this value and the outcome of the non-end  $G$  almost entirely determine the outcome of the sum. We give  $a + c$  a name.

**Definition 3.4.1.** *The tilt of a sum  $X \in \mathcal{A}_e$ , denoted  $\text{tilt}(X)$ , is the integer  $a + c$ , where  $X = aA + bB + cC$ .*

Note that  $\text{tilt}(X + Y) = \text{tilt}(X) + \text{tilt}(Y)$ , for any sums  $X$  and  $Y$  of alternating ends. The proofs of this section refer repeatedly to Theorem 3.2.10, which can be restated as follows using the notation of Definition 3.4.1:

$$o^-(X) = \begin{cases} \mathcal{L}^- & \text{if } \text{tilt}(X) < 0, \\ \mathcal{N}^- & \text{if } \text{tilt}(X) = 0, \\ \mathcal{R}^- & \text{if } \text{tilt}(X) > 0. \end{cases}$$

A game  $X = aA + bB + cC$  can be represented by  $(a, b, c)$ . To simplify the arguments below, some immediate observations about the left and right options from  $(a, b, c)$  are identified in Lemma 3.4.2. One of the most crucial points is that when Left can move in a sum of alternating ends, she can always decrease the tilt by one, and when Right has a move in a sum of ends, he can increase the tilt by one. This and the other claims of Lemma 3.4.2 can be confirmed immediately by reviewing Figure 3.7 of Section 3.2.

**Lemma 3.4.2.** *Let  $X = (a, b, c)$  be a sum of alternating ends, let  $X' = (a', b', c')$  be a general move from  $X$ , and let  $G$  be any alternating position.*

1. If Left has a move in  $X$  then  $\text{tilt}(X^L) = \text{tilt}(X) - 1$  for some Left option  $X^L$ , and  $\text{tilt}(X) - 1 \leq \text{tilt}(X^L) \leq \text{tilt}(X)$  for all Left options  $X^L$ .
2. If Right has a move in  $X$  then  $\text{tilt}(X^R) = \text{tilt}(X) + 1$  for some Right option  $X^R$ , and  $\text{tilt}(X) \leq \text{tilt}(X^R) \leq \text{tilt}(X) + 1$  for all Right options  $X^R$ .
3. If Left moves in  $X = (a, b, c)$  to  $X' = (a', b', c')$  then  $b - c - 1 \leq b' - c' \leq b - c$ .
4. If Right moves in  $X = (a, b, c)$  to  $X' = (a', b', c')$  then  $b - c \leq b' - c' \leq b - c + 1$ .
5. If either player can move so that  $b' - c' \neq b - c$ , then that player can move so that  $b' - c' \neq b - c$  and  $\text{tilt}(X) = \text{tilt}(X')$ .
6. If  $G \in \mathcal{L}^- \cup \mathcal{P}^-$  then  $\text{tilt}(G^R) = 0$  for every right option  $G^R$ .
7. If  $G \in \mathcal{R}^- \cup \mathcal{P}^-$  then  $\text{tilt}(G^L) = 0$  for every left option  $G^L$ .
8. If  $G \in \mathcal{L}^- \cup \mathcal{N}^-$  then  $\text{tilt}(G^L) = -1$  for some left option  $G^L$ .
9. If  $G \in \mathcal{R}^- \cup \mathcal{N}^-$  then  $\text{tilt}(G^L) = 1$  for some right option  $G^R$ .

We are about to determine the outcome of  $G + (a, b, c)$  of a single non-end alternating position  $G$ . Intuitively, if  $X$  is already tilted sufficiently in Left's favour (that is, if  $a + c$  is sufficiently less than zero), then adding a game  $G$  has little effect, and so Left wins  $G + X$ . Similarly, if  $\text{tilt}(X)$  is large enough, then Right will win  $G + X$  for any  $G$ . The work comes in showing what happens when the tilt is balanced or nearly balanced. If  $-1 \leq \text{tilt}(X) \leq 1$ , then the players suddenly care about their moves in  $G$ . If a player has a good move in  $G$ , he or she may be able to win on the sum, even if the tilt of  $X$  is zero or slightly in the opponent's favour. The outcome of  $G + X$  in each of these cases is stated in Theorems 3.4.3 and 3.4.3B.

**Theorem 3.4.3.** *If  $G$  is an alternating position and  $X = (a, b, c)$  is a sum of alternating ends, with  $t = a + c = \text{tilt}(X)$ , then*

$$o^-(G + X) = \begin{cases} \mathcal{L}^-, & t < -1, \\ & t = -1 \text{ and } G \in \mathcal{L}^- \cup \mathcal{N}^-, \\ & t = -1, b > c, \text{ and } G \in \mathcal{P}^-, \\ & t = 0 \text{ and } G \in \mathcal{L}^-, \text{ or} \\ & t = 0, b > c, \text{ and } G \in \mathcal{P}^-; \\ \mathcal{N}^-, & t = -1 \text{ and } G \in \mathcal{R}^-, \\ & t = -1, b \leq c, \text{ and } G \in \mathcal{P}^-, \\ & t = 0 \text{ and } G \in \mathcal{N}^-, \\ & t = 1 \text{ and } G \in \mathcal{L}^-, \text{ or} \\ & t = 1, b \geq c, \text{ and } G \in \mathcal{P}^-; \\ \mathcal{R}^-, & t > 1, \\ & t = 1 \text{ and } G \in \mathcal{R}^- \cup \mathcal{N}^-, \\ & t = 1, b < c, \text{ and } G \in \mathcal{P}^-, \\ & t = 0 \text{ and } G \in \mathcal{R}^-, \text{ or} \\ & t = 0, b < c \text{ and } G \in \mathcal{P}^-; \\ \mathcal{P}^-, & t = 0, b = c, \text{ and } G \in \mathcal{P}^-. \end{cases}$$

It is perhaps easier to see the conditions of Theorem 3.4.3 stated as separate results based on the outcome of  $G$ . It is certainly more natural to prove the theorem in this way. Thus, we prove the restated version Theorem 3.4.3B below.

**Theorem 3.4.3B.** *Let  $G$  be an alternating position and let  $X = (a, b, c)$  be a sum of alternating ends, with  $t = a + c = \text{tilt}(X)$ .*

1. *If  $G \in \mathcal{L}^-$  then*

$$o^-(G + X) = \begin{cases} \mathcal{L}^-, & t \leq 0, \\ \mathcal{N}^-, & t = 1, \\ \mathcal{R}^-, & t > 1. \end{cases}$$

2. If  $G \in \mathcal{R}^-$  then

$$o^-(G + X) = \begin{cases} \mathcal{L}^-, & t < -1, \\ \mathcal{N}^-, & t = -1, \\ \mathcal{R}^-, & t \geq 1. \end{cases}$$

3. If  $G \in \mathcal{N}^-$  then

$$o^-(G + X) = \begin{cases} \mathcal{L}^-, & t \leq -1, \\ \mathcal{N}^-, & t = 0, \\ \mathcal{R}^-, & t \geq 1. \end{cases}$$

4. If  $G \in \mathcal{P}^-$  then

$$o^-(G + X) = \begin{cases} \mathcal{L}^-, & t < -1, \text{ or } t = -1, 0 \text{ and } b > c; \\ \mathcal{N}^-, & t = -1 \text{ and } b \leq c, \text{ or } t = 1 \text{ and } b \geq c; \\ \mathcal{R}^-, & t > 1, \text{ or } t = 0, 1 \text{ and } b < c; \\ \mathcal{P}^-, & t = 0 \text{ and } b = c. \end{cases}$$

*Proof.* Although the result holds for any alternating position  $G$ , we may as well assume  $G$  is not an end, since otherwise the theorem reduces to Theorem 3.2.10. We first deal with  $t = \text{tilt}(X) < -1$  and  $t > 1$ , and then consider the four cases of  $o^-(G)$  for  $-1 \leq t \leq 1$ .

If  $t = \text{tilt}(X) < -1$  then Left wins first by moving to any  $G^L + X$ , since  $\text{tilt}(G^L) \in \{0, -1\}$  gives  $\text{tilt}(G^L + X) < -1$ , which shows by induction that  $G^L + X \in \mathcal{L}^-$ . If Right moves first to  $G + X^R$  then  $\text{tilt}(X^R) \leq -1$  and Left wins by induction (since in this case  $G + X^R \in \mathcal{L}^- \cup \mathcal{N}^-$ ); if Right moves first to  $G^R + X$  then  $\text{tilt}(G^R) \in \{0, 1\}$  means  $\text{tilt}(G^R + X) \leq -1$ , and then Left wins as the next player on the sum of ends. Similarly, if  $t > 1$  then Right wins playing first or second.

If  $G \in \mathcal{L}^-$  and  $t \in \{-1, 0\}$  then Left playing first moves to  $G^L + X$ , since  $\text{tilt}(G^L) = -1$  gives  $\text{tilt}(G^L + X) \leq -1$ , which means  $G^L + X$  is a left-win sum of ends. If  $G \in \mathcal{L}^-$  and  $t = 1$  then Left has a move in  $X$  and so can reduce the tilt by one, leaving  $G + X^L$  with  $\text{tilt}(X^L) = 0$ , which is left-win by induction. Thus Left wins playing first in each case. When  $t = 1$  Right has a good first move to  $G^R + X$  for any  $G^R$ , since  $\text{tilt}(G^R) = 0$  gives  $\text{tilt}(G^R + X) = 1$ , which means  $G^R + X$  is a

right-win sum of ends. In the other two cases,  $t = 0$  and  $t = -1$ , Right has no good first move: moving in  $G$  leaves a sum of ends with negative or zero tilt, and playing in  $X$  loses by induction. Thus  $G + X \in \mathcal{L}^-$  if  $G \in \mathcal{L}^-$  and  $t \in \{-1, 0\}$ , and  $G + X \in \mathcal{N}^-$  if  $G \in \mathcal{L}^-$  and  $t = 1$ . The case  $G \in \mathcal{R}^-$  follows by symmetry.

If  $G \in \mathcal{N}^-$  then Left wins when  $t = -1$  by the same reasoning as above:  $G^L + X \in \mathcal{L}^-$  because  $\text{tilt}(G^L + X) < -1$ , and the right moves  $G^R + X$ ,  $G + X^R$  are losing because the first is a sum of ends with nonpositive tilt, and the second is next-win by induction ( $G \in \mathcal{N}^-$ ,  $\text{tilt}(X^R) \leq 0$ ). By symmetry, Right wins playing first or second when  $t = 1$ . If  $t = 0$  then both players have a good first move, playing in  $G$  to a sum of ends tilted in their favour. Thus, if  $G \in \mathcal{N}^-$ , then  $G + X \in \mathcal{L}^-, \mathcal{N}^-, \mathcal{R}^-$  when  $t = -1, 0, 1$ , respectively.

If  $G \in \mathcal{P}^-$  then there is slightly more to show. Let  $X' = (a', b', c')$  be a general Left or Right option from  $X = (a, b, c)$ . Recall from Lemma 3.4.2 that a Left move in  $X$  either leaves the value of  $b - c$  unchanged or decreases it by one, so that if  $b > c$  then  $b' \geq c'$ . If Right moves in  $X$  then the value of  $b - c$  is either the same or increased by 1. Furthermore, for both players, if there is a move that changes  $b - c$  then there is a move that changes  $b - c$  while not changing the tilt,  $a - c$ .

If  $t = 0$  and  $b > c$ , then Left must have a move in  $X$ ; otherwise  $b$  must be negative, forcing  $c$  to be negative, and then  $a$  must be positive to have  $a + c = 0$ , so Left would have a move in an  $A$ . So Left can move in  $X$  to  $X' = (a', b', c')$  with  $b' \geq c'$  and  $\text{tilt}(X') \leq \text{tilt}(X)$ . If  $b' > c'$  then  $\text{tilt}(X) \in \{0, -1\}$  means  $X'$  is in  $\mathcal{L}^-$  by induction; if  $b' = c'$  then  $\text{tilt}(X') = \text{tilt}(X) = 0$  means  $X'$  is in  $\mathcal{P}^-$  by induction. Thus, Left has a good first move in  $G + X$  when  $t = 0$  and  $b > c$ . On the other hand, Right loses  $G + X$  if he moves in  $G$ , since  $\text{tilt}(G^R) = 0$  leaves a sum of ends with zero tilt, which is next-win. Right also loses moving in  $X$ : no Right move can decrease  $b - c$ , so  $G + X'$  has  $b' > c'$  and  $\text{tilt}(X') \leq 1$ , which by induction means  $G + X'$  is left-win or next-win. Thus  $G \in \mathcal{P}^-, t = 0, b > c$  gives  $G + X \in \mathcal{L}^-$ .

By symmetry, if  $t = 0$  and  $b < c$  then  $G + X \in \mathcal{R}^-$ . If  $t = 0$  and  $b = c$  then neither player has a good first move: Left cannot move to  $b' > c'$ , Right cannot move to  $b' < c'$ , and neither can move to  $b' = c'$  with  $t = 0$ , as the only move that preserves  $t = 0$  is a move in  $B$  or  $\bar{B}$  to zero, which has  $b' = b \pm 1, c' = c$ , so that  $b' \neq c'$ .

If  $G \in \mathcal{P}^-$  and  $t = -1$  then Left has a good first move to  $G^L + X$ . When  $b > c$ ,

Right has no good first move, since moving in  $G$  leaves negative tilt and moving in  $X$  leaves  $b' > c', t \leq 0$ , which is left-win by induction. So  $G + X \in \mathcal{L}^-$  when  $G \in \mathcal{P}^-$ ,  $t = -1$ , and  $b > c$ ; by symmetry,  $G + X \in \mathcal{R}^-$  when  $G \in \mathcal{P}^-$ ,  $t = 1$ , and  $b < c$ . If  $t = -1$  (so there is either an  $\bar{A}$  or  $\bar{C}$ ) and  $b \leq c$ , Right can win first playing in  $\bar{A}$  or  $\bar{C}$  to  $t' = 0, b' \leq c'$ , which is either right-win ( $b' < c'$ ) or previous-win ( $b' = c'$ ), by induction. Thus  $G + X \in \mathcal{N}^-$  when  $t = -1$  and  $b \leq c$ , or, by symmetry, when  $t = 1$  and  $b \geq c$ .  $\square$

The trouble with inductive arguments like the one above is that, though quite effective, they hide all of the intuition behind a result. Why is it that when  $G \in \mathcal{P}^-$ , the value of  $b - c$  arises as one of the determinants of the outcome of  $G + (a, b, c)$ ? To understand this aspect of alternating sums, consider the following example, illustrated in Figure 3.10. Let  $G = *$  and  $X = B$ , so that  $G \in \mathcal{P}^-$ ,  $t = 0$ , and  $b > c$ . Left playing first should move in  $B$  to 0 in order to force Right to play in  $*$ . This is the first instance of Left winning with a counterintuitive move: generally Left and Right play in  $(a, b, c)$  to decrease and increase the tilt respectively. Maintaining the tilt is a good move for Left when each player is trying to avoid playing first on the previous-win position. However, this option is only available to Left if there are more  $B$  positions than  $C$  positions, because every  $C$  will open up a  $\bar{B}$  in which Right can similarly choose to maintain the tilt ( $\bar{B} \rightarrow 0$ ), thereby cancelling the effect of Left's move. For this reason the value of  $b - c$  affects the outcome of  $G + X$  when  $G \in \mathcal{P}^-$ .

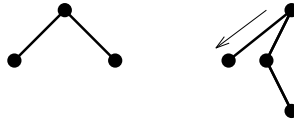


Figure 3.10: An example of Left preferring the move  $B \rightarrow 0$  over the move  $B \rightarrow \bar{A}$ .

If  $G$  is not a previous-win position, then Theorem 3.4.3 shows  $a + c$  and  $o^-(G)$  are the only determinants of  $o^-(G + X)$ . Thus, in a sum of one alternating non-end  $G \notin \mathcal{P}^-$  and multiple alternating ends,  $B$  and  $\bar{B}$  are irrelevant; they are equivalent to zero in the sum. The move to  $\bar{A}$  or  $A$  dominates the move to 0, so that  $B$  is equivalent to  $\{\bar{A} | \cdot\}$ , which is in turn equivalent to 0 (because it is in  $\mathcal{N}^- \cap \mathcal{P}^+$ ). It seems that Left moving  $B$  to 0 instead of  $\bar{A}$  — that is, Left choosing not to change

the tilt in her favour — can only be a good move when playing with a  $\mathcal{P}^-$  position. It is reasonable to conjecture that in sums of *multiple* non-ends and multiple ends, if there are no previous-win positions, then Left and Right always choose to decrease and increase the tilt, respectively, so that  $B$  and  $\overline{B}$  are equivalent to zero. As in  $\mathcal{A}_e$ , this would further imply that  $C \equiv A$  and  $\overline{C} \equiv \overline{A}$ , modulo a universe of no previous-win positions. This idea has not been further explored and so is presented here as a conjecture.

**Conjecture 3.4.4.** *If  $(a, b, c)$  is a sum of alternating ends and  $Y$  is a sum of alternating positions, none of which are in  $\mathcal{P}^-$ , then  $(a, b, c) + Y \equiv (a + c)A + Y$ .*

Conjectures aside, there are some immediate uses for Theorem 3.4.3. As shown below, it can be used to prove that any alternating position in  $\mathcal{L}^- \cup \mathcal{N}^- \cup \mathcal{R}^-$  has an additive inverse in  $\mathcal{A}$ . By Theorem 2.4.2, proving  $G + \overline{G} \equiv 0$  can be reduced to showing that Left wins playing first on  $G + \overline{G} + X$ , for a left end  $X$ . From such a position, Left can play in  $G$  or  $\overline{G}$  to obtain a sum of one non-end and multiple ends, and then Theorem 3.4.3 can be applied.

**Theorem 3.4.5.** *If  $G \notin \mathcal{P}^-$  is an alternating position then  $G + \overline{G} \equiv 0 \pmod{\mathcal{A}}$ .*

*Proof.* Let  $S$  be the set of alternating positions in  $\mathcal{L}^- \cup \mathcal{R}^- \cup \mathcal{N}^-$ . Since  $S$  is closed under conjugates, Theorem 2.4.2 applies: we need only show that  $G + \overline{G} + X \in \mathcal{L}^- \cup \mathcal{N}^-$  for every  $G \in S$  and every left end  $X \in \mathcal{A}$ . Note that  $X$  is an arbitrary sum of alternating lefts ends, but may not itself be alternating.

We need to show Left wins playing first on  $G + \overline{G} + X$ . Since Theorem 3.2.9 has already established the invertibility of ends, we can assume  $G$  is not an end. Either  $G \in \mathcal{R}^-$  and  $\overline{G} \in \mathcal{L}^-$ , or both games are in  $\mathcal{N}^-$ . If the former, Left can win  $G + \overline{G} + X$  playing first to  $G^L + \overline{G} + X$ , since  $\text{tilt}(G^L) = 0$  and  $\text{tilt}(X) \leq 0$  means  $\overline{G} + G^L + X = \overline{G} + X'$  has  $\overline{G} \in \mathcal{L}^-$  and  $\text{tilt}(X') \leq 0$ , and so is left-win by Theorem 3.4.3. If the latter, then Left can move in  $G \in \mathcal{N}^-$  to decrease the tilt, leaving  $\overline{G} + X'$  with  $\overline{G} \in \mathcal{N}^-$  and  $\text{tilt}(X') < 0$ , which is also left-win by Theorem 3.4.3.  $\square$

Note that Theorem 3.4.5 does indeed require  $G \notin \mathcal{P}^-$ . If  $G$  is any one of the four previous-win positions then  $G + \overline{G}$  is not equivalent to zero in  $\mathcal{A}$ . It is already known that  $* + * \not\equiv 0 \pmod{\mathcal{A}}$ , because  $A = 1 \in \mathcal{A}$ . The sum  $\{0 \mid B\} + \{\overline{B} \mid 0\}$  is also

distinguished from zero by  $A$ , since  $\{0 \mid B\} + \{\overline{B} \mid 0\} + A \in \mathcal{P}^-$  while  $A \in \mathcal{R}^-$ . Finally, the position  $\{\overline{B} \mid B\}$  is as far from being invertible as can be:  $\{\overline{B} \mid B\} + \{\overline{B} \mid B\} \in \mathcal{P}^-$  is not even next-win.

The results of this section are really just preliminary in the understanding of the entire universe  $\mathcal{A}$ . We might conjecture that the outcome of  $X + Y$ , with  $X$  a sum of ends and  $Y$  an arbitrary sum of non-ends, will be primarily dependant on the tilt of  $X$  and the individual outcomes of the positions in  $Y$ . If  $\text{tilt}(X)$  is small enough, then nothing Right can do will tilt the sum back in his favour; similarly, Left will lose  $X + Y$  if  $\text{tilt}(X)$  is sufficiently large. If the tilt is ‘near’ zero (which at the very least would include the interval from  $-|Y|$  to  $|Y|$ ), then the complexity and tediousness of a case-by-case analysis can be extrapolated from Theorem 3.4.3. Thus, a complete description of the monoid of  $\mathcal{A}$  has not been attempted. Fortunately, however, with only what we do know about sums of alternating positions, we are able to successfully analyze a naturally-occurring subuniverse of  $\mathcal{A}$ : positions of the game PENNY NIM.

### 3.5 PENNY NIM

Recall the game PENNY NIM as described in Section 3.1. A position looks like some number of coin stacks, with each stack having all coins heads up or all coins tails up. Left plays by choosing any tails-up stack, removing at least one coin, and inverting the remaining coins; Right plays similarly on any heads-up stack. This game was introduced as an example of alternating *ends*, but we can make non-end positions by allowing a stack to initially be ‘sideways’, so that either player can make the first move. A player who removes one or more coins from a sideways stack orients any remaining coins appropriately, so that he or she cannot make two consecutive moves on the same stack.

Denote the closure of all PENNY NIM positions by  $\mathcal{P}$ . As claimed in Section 3.2, upright stacks are all equivalent to  $A$ ,  $B$ , or their conjugates. A single coin is exactly  $A$  or  $\overline{A}$ , and a stack of two coins — having options to zero and  $\overline{A}$  or  $A$  — is either  $B$  or  $\overline{B}$ . A tails-up stack of three coins has options to zero,  $\overline{A}$ , and  $\overline{B}$ , but this reduces to  $\{0, \overline{A} \mid \cdot\} = B$ , since by Theorem 3.2.5 the option to zero dominates the option to  $\overline{B}$ . An identical argument shows that a tails-up stack of more than three coins is also equivalent to  $B$ , as shown below.

**Lemma 3.5.1.** *If  $G$  is a tails-up stack of 3 or more coins then  $G \equiv B \pmod{\mathcal{P}}$ .*

*Proof.* We have already seen that a stack of one coin is  $A$  or  $\bar{A}$  and a stack of two coins is  $B$  or  $\bar{B}$ . Assume any stack of more than two but less than  $n$  coins is equivalent to  $B$  or  $\bar{B}$ . A stack of  $n$  coins then has options to  $0$ ,  $\bar{A}$ , and  $\bar{B}$ , which by Theorem 3.2.5 reduces to  $\{0, \bar{A} \mid \cdot\} = B$ .  $\square$

This shows that the ends in  $\mathcal{P}$  do not include  $C$ . Next, consider the possible non-end positions in this generalized version of PENNY NIM. A sideways stack of one coin has exactly one option,  $0$ , for both players, and so is the position  $*$ . A sideways stack of two coins is the position  $\{0, \bar{A} \mid 0, A\}$ , and a sideways stack of two or more coins is, by Lemma 3.5.1, equivalent to the same. Let us call this next-win position ‘ $D$ ’:

$$D = \{0, \bar{A} \mid 0, A\}.$$

The universe of PENNY NIM positions is the closure of  $\{0, A, B, D, *\}$ . A general position in  $\mathcal{P}$  is of the form  $aA + bB + dD + k*$  for integers  $a, b, d, k$ , but in fact we can do better than this. By Theorem 3.4.5, since  $D \notin \mathcal{P}^-$ , we know  $D + \bar{D} = D + D \equiv 0 \pmod{\mathcal{A}}$ . Thus, the coefficient  $d$  in the general sum is restricted to  $\{0, 1\}$ . We cannot say the same for  $*$ , as it is one of the four alternating positions that does not have an additive inverse in the larger universe (nor in the universe  $\mathcal{P} \subset \mathcal{A}$ , since  $A \in \mathcal{A}$ ). However, since  $\bar{*} = *$ , we can assume  $k \geq 0$ . Our goal is to determine the outcome of the game  $aA + bB + dD + k*$  where  $a, b \in \mathbb{Z}$ ,  $d \in \{0, 1\}$ , and  $k \in \mathbb{Z}^+$ . This result is presented as Theorem 3.5.6.

To find the outcome of  $aA + bB + dD + k*$ , we must extend slightly the main result of the previous section (Theorem 3.4.3), which gives the outcome of one non-end and a sum of ends. Here we have one non-end ( $D$ ) added to a sum of ends and any number of  $*$  positions. The following series of lemmas serve as base cases for the main inductive proof of Theorem 3.5.6. The first of these shows that the outcome of any multiple of the position  $*$  depends entirely on the parity. The second and third follow directly from Theorem 3.4.3.

**Lemma 3.5.2.** *For any  $k \in \mathbb{Z}$ ,*

$$o^-(k \cdot *) = \begin{cases} \mathcal{N}^- & \text{if } k \text{ is even,} \\ \mathcal{P}^- & \text{if } k \text{ is odd.} \end{cases}$$

*Proof.* It suffices to consider  $k \geq 0$ , since  $-* \equiv * \pmod{\mathcal{P}}$ . The result holds when  $k = 0, 1$ . If  $k > 1$  is even then either player can win first moving to  $(k-1) \cdot * \in \mathcal{P}^-$ , and if  $k > 1$  is odd then the only option for both players is  $(k-1) \cdot * \in \mathcal{N}^-$ .  $\square$

**Lemma 3.5.3.**

$$o^-(D + aA + bB) = o^-(aA + bB) = \begin{cases} \mathcal{L}^- & \text{if } a \leq -1, \\ \mathcal{N}^- & \text{if } a = 0, \\ \mathcal{R}^- & \text{if } a \geq 1. \end{cases}$$

*Proof.*  $D + aA + bB = G + X$  where  $G = D \in \mathcal{N}^-$  and  $\text{tilt}(X) = a$ . The result follows directly from Theorem 3.4.3B (part 3).  $\square$

**Lemma 3.5.4.**

$$o^-( * + aA + bB ) = \begin{cases} \mathcal{L}^-, & \text{if } a < -1 \text{ or } -1 \leq a \leq 0, b > 0; \\ \mathcal{N}^-, & \text{if } a = -1, b \leq 0 \text{ or } a = 1, b \geq 0; \\ \mathcal{R}^-, & \text{if } a > 1 \text{ or } 0 \leq a \leq 0, b < 0. \\ \mathcal{P}^-, & \text{if } a = 0 = b. \end{cases}$$

*Proof.*  $* + aA + bB = G + X$  where  $G = * \in \mathcal{P}^-$ ,  $\text{tilt}(X) = a$ , and  $c = 0$ . The cases follow directly from Theorem 3.4.3B (part 4).  $\square$

The final lemma is not so immediate. Here we consider the outcome of a single  $*$  position with  $D + aA + bB$ . Since many of the left and right options are of the form handled by Theorem 3.4.3, we rely heavily on that theorem, as well as the specific cases presented in the previous lemmas, to determine the outcome of  $* + D + aA + bB$ .

**Lemma 3.5.5.**

$$o^-( * + D + aA + bB ) = \begin{cases} o^-( * + aA + bB ) & \text{if } a, b \neq 0, \\ \mathcal{N}^- & \text{if } a = 0 = b. \end{cases}$$

*Proof.* It will be useful to identify the right options from  $G = * + D + aA + bB$  :

$$G^{R_1} = D + aA + bB;$$

$$G^{R_2} = * + aA + bB;$$

$$G^{R_3} = * + (a + 1)A + bB;$$

$$G^{R_4} = * + D + (a + 1)A + bB, \text{ if } a < 0;$$

$$G^{R_5} = * + D + (a + 1) + (b + 1)B, \text{ if } b < 0;$$

$$G^{R_6} = * + D + aA + (b + 1)B, \text{ if } b < 0.$$

If  $a \leq -1$ , Left wins playing first by moving  $*$  to 0, since in this case  $D + aA + bB \in \mathcal{L}^-$ . If  $a < -1$  then Right has no good first move:  $G^{R_1}, G^{R_2} \in \mathcal{L}^-$ ;  $G^{R_3} \in \mathcal{L}^- \cup \mathcal{N}^-$ ; and by induction,  $G^{R_4}, G^{R_5} \in \mathcal{L}^- \cup \mathcal{N}^-$ ,  $G^{R_6} \in \mathcal{L}^-$ . If  $a = -1$  and  $b \geq 1$ , the first four options remain bad for Right, while the last two do not exist; however, if  $a = -1$  and  $b \leq 0$ , Right has a good first move to  $G^{R_3} \in \mathcal{P}^-$  (if  $b = 0$ ) or  $G^{R_5} \in \mathcal{R}^-$  (if  $b \leq -1$ ).

By symmetry, this proves all cases besides  $a = 0$ . In this case, if  $b = 0$  we can check by hand that  $* + D \in \mathcal{N}^-$ . If  $b > 0$ , Left has a good first move in  $D$  to  $\bar{A}$ , leaving  $* + (a - 1)A + bB \in \mathcal{L}^-$ , while all Right options are bad or nonexistent:  $G^{R_1} \in \mathcal{N}^-$ ,  $G^{R_2} \in \mathcal{L}^-$ ,  $G^{R_3} \in \mathcal{N}^-$ , and  $G^{R_4}, G^{R_5}, G^{R_6}$  do not exist. By symmetry the outcome is  $\mathcal{R}^-$  when  $a = 0$  and  $b < 0$ .  $\square$

Notice that the presence of a  $D$  position only affects the outcome of the sum if  $a$  and  $b$  are both zero; that is, if the position is  $D + *$  as opposed to  $*$ . This makes some intuitive sense based on our previous observations, as in almost all cases, an option to  $A$  or  $\bar{A}$  dominates an option to zero, so that  $D = \{0, \bar{A} \mid 0, A\} \approx \{\bar{A} \mid A\} \equiv 0$  is often irrelevant. This trend continues when multiple  $*$  positions are present, as demonstrated in the main result below. Again we see that when the tilt of the alternating ends is sufficiently small, a sum of alternating positions is left win. The threshold is  $\pm k$  for  $k$  the number of  $*$  positions: essentially, if the tilt is less than  $-k$ , then Left can simply kill off all the  $*$  positions before Right has time to increase the tilt to nonnegative. Similarly, Right wins if the tilt is sufficiently large. When the tilt is somewhere in between  $-k$  and  $k$ , other factors will sway the outcome: namely, the presence of a  $D$  position and the value of  $b$ .

**Theorem 3.5.6.** For any  $k \in \mathbb{Z}^+$  and  $d \in \{0, 1\}$ ,

$$o^-(k * +dD + aA + bB) = \begin{cases} \mathcal{L}^-, & \text{if } a < -k \text{ or } -k \leq a < k, b \geq 1; \\ \mathcal{N}^-, & \text{if } a = -k, b \leq 0, \text{ or } a = k, b \geq 0, \\ & \text{or } -k < a < k, b = 0, a \equiv k \pmod{2}, d = 0, \\ & \text{or } -k < a < k, b = 0, d \neq 0; \\ \mathcal{R}^-, & \text{if } a > k \text{ or } -k < a \leq k, b \leq -1; \\ \mathcal{P}^-, & \text{if } -k < a < k, a \not\equiv k \pmod{2}, b = 0 = d. \end{cases}$$

*Proof.* Since we have already shown the cases  $k = 0, 1$ , we can assume  $k \geq 2$ . We have a similar set of options for Right as in the previous lemma. If  $G = k * +dD + aA + bB$  then the possible right options are enumerated below.

$$G^{R_1} = (k - 1) * +dD + aA + bB;$$

$$G^{R_2} = k * +aA + bB, \text{ if } d = 1;$$

$$G^{R_3} = k * +(a + 1)A + bB, \text{ if } d = 1;$$

$$G^{R_4} = k * +dD + (a + 1)A + bB, \text{ if } a < 0;$$

$$G^{R_5} = k * +dD + (a + 1) + (b + 1)B, \text{ if } b < 0;$$

$$G^{R_6} = k * +dD + aA + (b + 1)B, \text{ if } b < 0.$$

Let  $G^{L_1}, \dots, G^{L_6}$  be the symmetric moves for Left. If  $a < -k$ , Left wins first moving in  $*$  to  $G^{L_1}$ , which is in  $\mathcal{L}^-$  since  $a < -k < -k + 1 = -(k - 1)$ . Right has no good first move in this case:  $G^{R_1}$  and  $G^{R_2}$  are in  $\mathcal{L}^-$  while the remainder are in either  $\mathcal{L}^-$  or  $\mathcal{N}^-$ .

If  $a = -k$  then  $G^{L_1}$  is still a good first move for Left. If  $b = 0$  then Right has a good first move:  $G^{R_3} \in \mathcal{P}^-$  if  $b = 0, d \neq 0$ , and  $G^{R_4} \in \mathcal{P}^-$  if  $b = 0, d = 0$ , since in both cases the new coefficient of  $A$  is  $a + 1 \equiv k + 1 \pmod{2}$ . If  $b \leq -1$  then Right still has a good move:  $G^{R_4}$  is in  $\mathcal{R}^-$  in this case (as is  $G^{R_3}$ , if it exists). If  $b \geq 1$  then the first four right options are left-win positions and the last two do not exist.

If  $-k < a \leq 0$  then we break into cases based on the value of  $b$ . If  $b \geq 1$  then Left wins with a move to  $G^{R_1}$ , which is left-win by induction because we still have

$b \geq 1$ , and the new  $k$ -value,  $k' = k - 1$ , satisfies  $-k' \leq a$ . Right's possible moves are  $G^{R_1} = G^{L_1} \in \mathcal{L}^-$  as just argued;  $G^{R_2} \in \mathcal{L}^-$  because there is no change to  $a, b, k$ ; and  $G^{R_3} \in \mathcal{L}^-$ ,  $G^{R_4} \in \mathcal{L}^-$  because  $a' = a + 1$  satisfies  $-k < a < k'$ . The options  $G^{R_5}, G^{R_6}$  do not exist because  $b \geq 1$ . Thus Right has no good first move and we see the sum is left-win when  $-k < a \leq 0$  and  $b \geq 1$ .

Suppose  $-k < a \leq 0$  and  $b = 0$ . If  $d = 1$  then both players have a good first move to either  $G^{R_2} / G^{L_2}$  or  $G^{R_3} / G^{L_3}$ , whichever moves leaves the correct parity of  $a$  and  $k$  values. In particular, if  $a$  and  $k$  are of the same parity then a player should move in  $D$  to  $A$  or  $\bar{A}$  to switch the parity and leave a  $\mathcal{P}^-$  position; if  $a$  and  $k$  are not of the same parity then a player should move  $D$  to zero. If  $d = 0$  and  $a, k$  have the same parity then both players have a good move to  $G^{R_1} / G^{L_1}$ , since reducing the star count switches the parity of  $k$  and leaves a previous-win position. If  $d = 0$  and  $a, k$  do *not* have the same parity, then neither player has a good first move: moves 2, 3, 5, 6 are not available for either player, since  $d = 0$  and  $b = 0$ ; move 1 is a next-win position because the parity of  $k$  is switched; and move 4 is a next-win position because the parity of  $a$  is switched.

Finally, if  $-k < a \leq 0$  and  $b \leq -1$ , then Right has a good first move to  $G^{R_4}$ , which is right-win by induction because the new  $a$  value,  $a' = a + 1$ , satisfies  $-k < -k + 1 < a + 1 = a'$ . Left, however, has no good first move:  $G^{L_1}$  and  $G^{L_3}$  are in  $\mathcal{N}^-$  (if  $-k + 1 = a$ ) or  $\mathcal{R}^-$  (otherwise);  $G^{L_2} \in \mathcal{R}^-$ ; and the remaining moves are not available. Thus the sum is in  $\mathcal{R}^-$  in this case.

The arguments for  $a > 0$  follow by symmetry. □

We now know the outcome of any PENNY NIM position, with sideways stacks allowable. Moreover, the proof of Theorem 3.5.6 is constructive: it shows what a player's good move is when he or she has one. The monoid of PENNY NIM is

$$\mathcal{M}_{\mathcal{P}} = \langle 0, A, \bar{A}, B, \bar{B}, D, * \mid A + \bar{A} = B + \bar{B} = D + D = 0 \rangle,$$

with the outcome partition given implicitly by Theorem 3.5.6. Unlike many of the monoids in this thesis,  $\mathcal{M}_{\mathcal{P}}$  is a monoid that is not a group: the element  $*$  has no inverse.

We can now answer a question posed in Section 3.1: who wins the PENNY NIM sum shown in Figure 3.1? The black stack of size three is equivalent to  $B$  and the

white stack of size two is equivalent to  $\overline{B}$ ; thus, these stacks cancel in the sum. We are left with two sideways stacks: one is equivalent to  $D$ , and the other is  $*$ . By Theorem 3.5.6, the outcome of  $D + *$  is  $\mathcal{N}^-$ , and the good first move for either player is to remove both coins in the stack of size two, leaving  $*$  for the opponent.

### 3.6 Future Research in Alternating Games

Alternating games form a useful universe for misère analysis because of their strictly-controlled game trees. Although the monoid of the entire universe of  $\mathcal{A}$  may be out of reach, there may be other interesting subuniverses, in addition to ends  $\mathcal{A}_e$  and PENNY NIM positions  $\mathcal{P}$ , to explore.

As an example, consider an alternating variant of the game MANCALA<sup>3</sup>. While any stones remain on Left's side of the board, Left must continue playing, and likewise Right can play if and only if there are stones on Right's side. Given the scarcity of alternating games, 'ALTERNATING MANCALA' is a relatively natural variation to consider. It would also be interesting to classify any non-alternating game that can be written as a disjunctive sum of alternating positions. For example, the game  $\{* \mid *\}$ , which is not itself alternating, can be decomposed into  $* + *$ , and so  $\{* \mid *\} \in \mathcal{A}$ . Likewise, any normal-play canonical form integer  $\mathbf{n} = \{\mathbf{n} - \mathbf{1} \mid \cdot\}$  is a sum of alternating ends:  $\mathbf{n} = \sum_{i=1}^{\mathbf{n}} A$ . Thus  $\mathbf{n} \in \mathcal{A}_e$ , and we know furthermore that  $\mathbf{n} + \overline{\mathbf{n}} \equiv 0 \pmod{\mathcal{A}}$ .

There are of course many positions that cannot be written as a sum of alternating games. Non-integer right ends whose followers are all right ends (that is, a game in which Right has no move now or later; in Chapter 5 these are called *dead right ends*), such as  $\{0, A \mid \cdot\}$ , would have to be a sum of alternating *ends*, and could moreover not be a sum of any  $B$  or  $C$  positions, since then Right would have a move in a follower. So such a game could only be a sum of  $A$  positions, but we just saw that a sum of  $A$  positions is an integer, and these ends are not integers. So non-integer dead right (and left) ends are not in  $\mathcal{A}$ . In general, how can we identify non-alternating positions which are actually sums of alternating components? Answering this question would potentially lead to many natural subuniverses of  $\mathcal{A}$  to which we could apply the knowledge of alternating games presented in this chapter.

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<sup>3</sup>Rules for MANCALA can be found in Appendix A.

## Chapter 4

### The Dicot Universe

#### 4.1 Introduction to Dicot Games

In normal play, a well-known class of games called *all-small* [5] are defined by the property that, at every position, either both players can move or neither player can. The name ‘all-small’ is given to such a game because it is necessarily *infinitesimal*: that is, it is less (under normal play) than every positive number, and greater than every negative number.

In misère play, all-small games form an interesting, additively closed universe in which many nice results arise; for example, Meghan Allen found that the day-1 game  $*$  =  $\{\cdot \mid \cdot\}$  is invertible here [3]. It is, however, no longer the case that these games are infinitesimal, and so a more appropriate misère-play name is given.

**Definition 4.1.1.** *A game  $G$  is a dicot if  $G = 0$  or if  $G^L$  and  $G^R$  are non-empty and every follower of  $G$  is a dicot.*

In botany, a *dicot* or *dicotyledon* is a flowering plant whose seed has two embryonic leaves (*cotyledons*); given that this loosely describes the game tree of a position satisfying Definition 4.1.1, the term is used in misère play in place of the term all-small. Let  $\mathcal{D}$  denote the set of all dicot positions.

In addition to the invertibility of  $*$  modulo  $\mathcal{D}$ , there have been some results for binary<sup>1</sup> dicot games [2]. In this chapter, the analysis of misère dicots is extended, beginning with several new general results in Section 4.2 and followed by an overview of dicots born by day 3 in Sections 4.3 and 4.4. In particular, these two sections establish the invertibility of all but one day-2 dicot and of a large number of day-3 dicots.

The monoid for the game of HACKENBUSH SPRIGS was developed during joint work with McKay and Nowakowski [9]; it is presented here in Section 4.5 as an application

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<sup>1</sup>A position  $G$  is binary if  $|G^L| \leq 1$ ,  $G^R \leq 1$ , and if every follower is also binary.

of the results of Section 4.2 and as an illustration of the techniques possible in the dicot universe. Finally, potential for further misère research in the dicot universe is discussed in the last section, 4.6.

## 4.2 General results for dicot games

This section presents several results that hold in the entire universe  $\mathcal{D}$ . As is common throughout this thesis, there is a focus here on equivalency to zero, either for a single game or for the sum of conjugate pairs. Most of these results prove useful in subsequent sections, as we analyze specific dicot positions or specific families of dicot positions.

The first task is to prove a claim of the previous section, that the universe  $\mathcal{D}$  is closed under the operation of disjunctive sum. This is obviously true in normal play, but that alone is not generally enough to guarantee the same for misère play; in Chapter 5, the set of normal-play numbers serves as an example. Theorem 4.2.1 settles the question for the present universe.

**Theorem 4.2.1.** *If  $G, H \in \mathcal{D}$  then  $G + H \in \mathcal{D}$ .*

*Proof.* If either  $G$  or  $H$  is zero then the result is trivial. Otherwise, the left and right options of both are non-empty, and both players have a move in  $G + H$ . Any follower of  $G + H$  is of the form  $G' + H'$  where  $G'$  and  $H'$  are (not necessarily proper) followers of  $G$  and  $H$ . If both are zero then their sum is zero, a dicot; if at least one, say  $G'$  is non-zero, then it is a dicot because  $G$  is, and so both players can move in  $G' + H'$ , as both can move in  $G'$ .  $\square$

Thus,  $\mathcal{D}$  is closed under disjunctive sum. It is also, by definition, closed under followers. These two properties are used implicitly in many subsequent arguments.

As a consequence of one of the general theorems of Chapter 2, we have the following sufficiency condition for invertible dicots, which makes demonstration of invertibility very straightforward in the universe  $\mathcal{D}$ . This theorem appears in [9].

**Theorem 4.2.2.** [9] *If  $G + \overline{G} \in \mathcal{N}^-$  and  $H + \overline{H} \in \mathcal{N}^-$  for all followers  $H$  of  $G \in \mathcal{D}$ , then  $G + \overline{G} \equiv 0 \pmod{\mathcal{D}}$ .*

*Proof.* Assume  $G + \overline{G} \in \mathcal{N}^-$  and that  $H + \overline{H} \in \mathcal{N}^-$  for all followers  $H$  of  $G$ . By induction we have  $H + \overline{H} \equiv 0 \pmod{\mathcal{D}}$  for all proper followers  $H$ . If  $Y$  is any left end in  $\mathcal{D}$ , then  $Y$  is necessarily the zero game  $\{\cdot | \cdot\}$ , and so we have  $G + \overline{G} + Y \equiv G + \overline{G} \in \mathcal{N}^-$ , which Left wins playing first. By Theorem 2.4.2, this gives  $G + \overline{G} \equiv 0 \pmod{\mathcal{D}}$ .  $\square$

The criteria of Theorem 4.2.2 is unfortunately not *necessary* for the invertibility of a dicot position. The game  $G = \{0 | 0, *_2\}$  is a counterexample to the converse, where  $*_2$  is the NIM heap of size two:  $*_2 = \{0, * | 0, *\}$ . We have  $*_2 + \overline{*}_2 = *_2 + *_2 \in \mathcal{P}^-$ , and so  $G = \{0 | 0, *_2\}$  has a follower  $H = *_2$  with  $H + \overline{H} \notin \mathcal{N}^-$ . However,  $G$  is nevertheless invertible, as shown in the proposition below.

**Proposition 4.2.3.** *The game  $G = \{0 | 0, *_2\}$  is invertible modulo  $\mathcal{D}$ ; that is*

$$\{0 | 0, *_2\} + \{0, *_2 | 0\} \equiv 0 \pmod{\mathcal{D}}.$$

*Proof.* Let  $X$  be any dicot game and suppose Left wins  $X$ . To win  $G + \overline{G} + X$ , Left follows her strategy from  $X$ . If Right plays in  $G$  or  $\overline{G}$  to 0, Left copies the move in  $\overline{G}$  or  $G$ , forcing Right to resume losing in  $X$ . If Right plays in  $G$  to  $*_2$  then Left responds there, moving to  $* + \overline{G} + X'$  (where  $X' \in \mathcal{L}^- \cup \mathcal{P}^-$ , since Right was to play next in  $X'$ ). Now, if Right ignores  $* + \overline{G}$  and plays only in  $X'$ , then Left runs out of moves first, plays in  $*$ , and wins when Right plays his only option of 0 in  $\overline{G}$ . Otherwise, Right plays in  $*$  and Left responds in  $\overline{G}$  to 0, or Right plays in  $\overline{G}$  and Left responds in  $*$  to 0; either way, Right is forced to resume play in  $X$ .

Thus, Right should not play at all in  $G$  or  $\overline{G}$ ; but then Left runs out of moves in  $X$  and wins playing first in  $G + \overline{G}$ , with a move to  $0 + \overline{G}$ . This shows (by symmetry) that the outcome of  $G + \overline{G} + X$  is the same as the outcome of  $X$ , and so  $G + \overline{G} \equiv 0 \pmod{\mathcal{D}}$ , as claimed.  $\square$

One of the main results of [3], that  $* + *$  is equivalent to zero in any universe of dicots, follows as a corollary to Theorem 4.2.2, since  $* + * \in \mathcal{N}^-$  and since the same is true for the only follower of  $*$ , zero.

**Corollary 4.2.4.** [3] *The game  $* = \{\cdot | \cdot\}$  satisfies  $* + * \equiv 0 \pmod{\mathcal{D}}$ .*

In the next two sections, Theorem 4.2.2 is similarly used to show that many of the dicot games born by day 3 have additive inverses modulo  $\mathcal{D}$ . Before turning to

those specific dicots, this section ends with a construction of a family of games that are equivalent to zero in this universe.

**Theorem 4.2.5.** *If  $G^{LR} = \{0\} = G^{RL}$  for every left option  $G^L$  and right option  $G^R$  of  $G \in \mathcal{D}$ , then  $G \equiv 0 \pmod{\mathcal{D}}$ .*

*Proof.* Let  $X$  be any dicot position and suppose Left wins  $X$ . Left can win  $G + X$  by playing as usual; if Right plays in  $G$  to  $G^R$  then Left responds to zero, forcing Right to resume losing in  $X$ . Once Left runs out of moves in  $X$  the position must be zero (as  $X$  is a dicot); Left can play to a  $G^L$ , from which Right's only option is zero, and so Left wins. A symmetric argument shows that Right wins  $G + X$  whenever he wins  $X$ , and so  $G \equiv 0 \pmod{\mathcal{D}}$ .  $\square$

Note that the position  $\{* | *\} = * + *$  is of the form described in Theorem 4.2.5, and so we have yet another result which gives  $* + * \equiv 0 \pmod{\mathcal{D}}$  as a corollary.

### 4.3 Day-2 Dicots

The first part of this section develops a base of knowledge about the dicot games born on day 2, by determining the seven distinct day-2 dicot positions, proving the invertibility of six of them, and establishing several instances of comparability. The scope is then narrowed to consider three inverse pairs in turn, as well as the single non-invertible position, finding the monoid of the closure of each.

With 0 and  $*$  as the only allowable options, there are eight (not necessarily distinguishable) new dicot games born on day 2. These are identified in Figure 4.1 below. When a position has the same game tree as (that is, is identical to) a named normal-play canonical form, the normal-play name is used, even when the rationale behind the notation fails in misère play. For example,  $\uparrow * = \{0 | 0, *\}$  is so-named because, in normal play, it is the canonical form of  $\uparrow + *$ ; this is not the case in misère play, but for consistency the name is still used.

Let us first determine whether or not zero and the nine games in Figure 4.1 are pairwise distinguishable modulo  $\mathcal{D}$ . Immediately we see that the game  $\{* | *\} = * + *$  is equivalent to zero, by Corollary 4.2.4. Among the other games,  $\uparrow$  and  $\downarrow *$  are right-win,  $\downarrow$  and  $\uparrow *$  are left-win, and  $E$ ,  $\overline{E}$ , and  $*_2$  are next-win. Two games with different outcomes are immediately distinguished; the bullet points below demonstrate that

$G^R$	0	*	0,*
0	$*$ 	$\uparrow$ 	$\downarrow*$ 
*	$\downarrow$ 	$* + *$ 	$\overline{E}$ 
0,*	$\uparrow*$ 	$E$ 	$*_2$ 

Figure 4.1: Non-zero dicot games born by day 2.

any two positions with the same misère outcome are also pairwise distinguishable, modulo  $\mathcal{D}$ .

- $\uparrow + \uparrow \in \mathcal{P}^-$  while  $\downarrow* + \uparrow \in \mathcal{R}^-$ .
- $\downarrow + \downarrow \in \mathcal{P}^-$  while  $\uparrow* + \downarrow \in \mathcal{L}^-$ .
- $E + * \in \mathcal{L}^-$ ,  $\overline{E} + * \in \mathcal{R}^-$ , and  $*_2 + * \in \mathcal{N}^-$ .

Thus, there are 7 distinct games born on day 2 in  $\mathcal{D}$ :  $\uparrow, \downarrow, \uparrow*, \downarrow*, E, \overline{E}$ , and  $*_2$ . It is easy to see that each of these, with the exception of  $*_2$ , is invertible modulo  $\mathcal{D}$ . Since the only followers of the games are 0 and  $*$  (which satisfy  $0 + 0 \in \mathcal{N}^-$  and  $* + * \in \mathcal{N}^-$ ), we need only check that  $G + \overline{G} \in \mathcal{N}^-$  for each day-2 dicot  $G \neq *_2$ , in order to apply Theorem 4.2.2. It is clear that  $*_2$  is not its own inverse, since  $*_2 + *_2 \in \mathcal{P}^-$  does not even have the same outcome as 0.

**Lemma 4.3.1.** *If  $G$  is a day-2 dicot and  $G \neq *_2$  then  $G + \overline{G} \equiv 0 \pmod{\mathcal{D}}$ .*

*Proof.* As established above, we need only show  $G + G \in \mathcal{N}^-$ , and then the result follows from Theorem 4.2.2. Left and Right can win  $\uparrow + \downarrow$  playing first, by taking their move to zero. Likewise both players can win first on  $\uparrow* + \downarrow*$ , with Left moving to  $\uparrow* + 0$  and Right moving to  $0 + \downarrow*$ . Finally, Left wins  $E + \overline{E}$  by moving to  $E + *$  and Right wins by moving to  $* + \overline{E}$ .  $\square$

In Section 4.4 the same technique is used to prove the invertibility (modulo  $\mathcal{D}$ ) of many day-3 dicots.

With distinguishability and invertibility already established, the final objective for the present is to consider comparability of the seven day-2 dicots. In fact, several pairs of these dicots are immediately comparable without restricting to  $\mathcal{D}$ , simply by using the ‘hand-tying’ principle — Lemma 2.4.1 in Chapter 2. Other comparisons can be established by using an equivalence modulo  $\mathcal{D}$  and then applying Lemma 2.4.1. The specific inequalities are outlined in Theorem 4.3.2 and illustrated in Figure 4.2

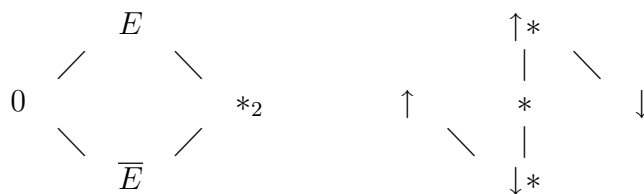


Figure 4.2: Some comparisons of day-2 dicots.

**Theorem 4.3.2.** *The following inequalities hold for day-2 dicot positions.*

1.  $\bar{E} \not\leq 0 \not\leq E \pmod{\mathcal{D}}$ ;
2.  $\bar{E} < *_2 < E$ ;
3.  $\downarrow * < * < \uparrow *$ ;
4.  $\downarrow * < \uparrow$ ;
5.  $\downarrow < \uparrow *$ .

*Proof.* We have already shown pairwise distinguishability, and so it suffices to show non-strict inequality in each case. By Lemma 2.4.1,  $G \leq H$  if  $G^R = H^R$  and  $\emptyset \neq G^L \subseteq H^L$ , or if  $G^L = H^L$  and  $G^R \supseteq H^R \neq \emptyset$ .

The position  $\bar{E} = \{ * | 0, * \}$  has one more right option than  $\{ * | * \} \equiv 0 \pmod{\mathcal{D}}$ , and so  $\bar{E} \leq 0 \pmod{\mathcal{D}}$ . Similarly,  $E = \{ 0, * | * \}$  has an additional left option over  $\{ * | * \}$ , so  $0 \equiv \{ * | * \} \leq E \pmod{\mathcal{D}}$ .

For the remaining inequalities, no restriction to the dicot universe is required. The position  $*_2 = \{ 0, * | 0, * \}$  is  $\bar{E} = \{ * | 0, * \}$  with an extra left option and  $E = \{ 0, * | * \}$

with an extra right option, so  $\overline{E} < *_2 < E$ . The same argument holds for  $* = \{0 \mid 0\}$  and the games  $\downarrow * = \{0 \mid 0, *\}$  and  $\uparrow * = \{0, * \mid 0\}$ . Similarly,  $\downarrow *$  has an extra right option over  $\uparrow = \{0 \mid *\}$ , so that  $\downarrow * < \uparrow$ , and  $\uparrow *$  has an extra left option over  $\downarrow = \{*\mid 0\}$ , so that  $\downarrow < \uparrow *$ .  $\square$

The following subsections focus on each pair of inverse day-2 dicots, and the exceptional  $*_2$ , determining the outcome of a general element of the closure of the games. That is, we find the monoids for  $cl(\uparrow, \downarrow)$ ,  $cl(\uparrow *, \downarrow *)$ ,  $cl(E, \overline{E})$ , and  $cl(*_2)$ . The first and last of these have previously been described [2, 15], but are included here for completeness. Note that the monoid of the closure of the single day-1 dicot,  $*$ , follows easily from Corollary 4.2.4:  $\mathcal{M}_* = \langle e, a \mid a^2 = e \rangle$ , with  $\mathcal{N}^- = \{e\}$  and  $\mathcal{P}^- = \{a\}$  [2]. In the arguments below,  $\delta$  is used as the coefficient of  $*$ , with  $\delta \in \{0, 1\}$ . The value  $\delta \pm 1$  is implied to be taken modulo 2.

#### 4.3.1 $cl(\uparrow, \downarrow)$

The closure of  $\uparrow$  and  $\downarrow$  is the set of all sums of  $\uparrow, \downarrow$ , and their follower  $*$ . We already know that  $\uparrow + \downarrow \equiv 0 \pmod{\mathcal{D}}$  and  $* + * \equiv 0 \pmod{\mathcal{D}}$ . These equivalencies also hold in any subuniverse of dicots; in particular, they hold in  $cl(\uparrow, \downarrow)$ . A general element of this closure is thus  $k \uparrow + \delta *$ , where  $k$  is any integer and  $\delta$  is 0 or 1. If  $k < 0$  then  $k \uparrow$  represents  $|k|$  copies of  $\downarrow$ .

The following lemma gives the outcome of  $k \uparrow + \delta *$  for  $k > 0$ . The outcomes for  $k < 0$  follow by symmetry (and if  $k = 0$  then the position is either 0 or  $*$ ).

**Lemma 4.3.3.** *If  $k$  is a positive integer and  $\delta \in \{0, 1\}$ , then*

$$o^-(k \uparrow + \delta *) = \begin{cases} \mathcal{L}^-, & \text{if } k \geq 4, \text{ or } k = 3, \delta = 0; \\ \mathcal{N}^-, & \text{if } k \leq 3, \delta = 1; \\ \mathcal{P}^-, & \text{if } k = 2, \delta = 0; \\ \mathcal{R}^-, & \text{if } k = 1, \delta = 0. \end{cases}$$

*Proof.* If we represent  $k \uparrow + \delta *$  as a pair  $(k, \delta)$ , then when  $k > 0$ , Left has a move in an  $\uparrow$  to  $(k - 1, \delta)$  and Right has a move in an  $\uparrow$  to  $(k - 1, \delta + 1)$ . If  $\delta = 1$  then both players have a move to  $(k, \delta - 1)$ .

If  $k \geq 4$  and  $\delta = 1$  then Left moves to  $(k, \delta - 1)$ , which is in  $\mathcal{L}^-$  by induction; if  $k \geq 4$  and  $\delta = 0$  then Left moves to  $(k - 1, 0)$ , which is in  $\mathcal{L}^-$  since  $k - 1 \geq 3$ . In either case Right has no good move:  $(k - 1, \delta + 1)$  is in either  $\mathcal{L}^-$  or  $\mathcal{N}^-$ , and as we have already seen,  $(k, \delta - 1) \in \mathcal{L}^-$ . If the position is  $(3, 0)$  then Left moves to  $(2, 0) \in \mathcal{P}^-$  and Right's only move is to  $(2, 1) \in \mathcal{N}^-$ . This shows the outcome is left-win if  $k \geq 4$  or if  $k = 3$  and  $\delta = 0$ .

Left wins  $(3, 1)$  by moving to  $(3, 0) \in \mathcal{L}^-$ , while Right wins the same position by moving an  $\uparrow$  to a  $*$ , cancelling the first  $*$ , leaving  $(2, 0) \in \mathcal{P}^-$ . Both players win  $(2, 1)$  by moving to  $(2, 0) \in \mathcal{P}^-$ . Left wins  $(1, 1)$  by moving to  $(0, 1) = * \in \mathcal{P}^-$  and Right wins  $(1, 1)$  by moving to  $(1, 0) \in \mathcal{R}^-$ .

Neither player has a good first move if the position is  $(2, 0)$ : Left's move to  $(1, 0)$  is right-win, and Right's move to  $(1, 1)$  is next-win. Finally, Right wins  $(1, 0)$  moving to  $(0, 1) = *$ , while Left's only move is to  $(0, 0) \in \mathcal{N}^-$ .  $\square$

The next two results show that for distinct values of  $k$  and  $j$ , the pairs  $k \uparrow$  and  $j \uparrow$ ,  $k \uparrow + *$  and  $j \uparrow + *$ , and  $k \uparrow$  and  $j \uparrow + *$ , are distinguishable. That is, every unique pair  $(k, \delta)$  gives a unique element in  $cl(\uparrow, \downarrow)$ .

**Lemma 4.3.4.** *If  $k \neq j$  are integers and  $\delta \in \{0, 1\}$ , then  $k \uparrow + \delta * \not\equiv j \uparrow + \delta * \pmod{\mathcal{D}}$ .*

*Proof.* First note that  $k \uparrow \not\equiv j \uparrow$  implies  $k \uparrow + * \not\equiv j \uparrow + *$ , since if  $o^-(k \uparrow + X) \equiv o^-(j \uparrow + X)$  then  $X + *$  distinguishes  $k \uparrow + *$  and  $j \uparrow + *$ . Thus, we need only distinguish  $k \uparrow$  and  $j \uparrow$ . We know  $j \uparrow + j \downarrow \equiv 0$  is next-win; by Lemma 4.3.3,  $k \uparrow + j \downarrow = (k - j) \uparrow$  (or  $(j - k) \downarrow$ ) cannot be next-win. This shows  $k \uparrow \not\equiv j \uparrow \pmod{\mathcal{D}}$  when  $k \neq j$ .  $\square$

**Lemma 4.3.5.** *If  $k$  and  $j$  are integers then  $k \uparrow + * \not\equiv j \uparrow \pmod{\mathcal{D}}$ .*

*Proof.* The outcome of  $k \uparrow + * + (3 - k) \uparrow \equiv 3 \uparrow + *$  is  $\mathcal{N}^-$ , while the outcome of  $j \uparrow + (3 - k) \uparrow \equiv (j - k + 3) \uparrow$  cannot be  $\mathcal{N}^-$ , by Lemma 4.3.3, unless  $j - k + 3 = 0$ . In that case we have  $j = k - 3$  and the games are distinguished instead by  $(-j + 1) \uparrow$ , since  $k \uparrow + * + (-k + 3 + 1) \uparrow \equiv 4 \uparrow + * \in \mathcal{L}^-$  while  $j \uparrow + (-j + 1) \uparrow \equiv \uparrow \in \mathcal{R}^-$ .  $\square$

The previous two lemmas show that the quotient group of  $cl(\uparrow, \downarrow)$  is  $\mathbb{Z} \times \mathbb{Z}_2$ , since every unique pair  $(k, \delta)$  generates a unique element  $k \uparrow + \delta *$ , modulo the closure.

Together with the outcome partition given by Lemma 4.3.3, we have found the misère monoid of this set to be

$$\mathcal{M}_{cl(\uparrow, \downarrow)} = \langle 0, *, \uparrow, \downarrow \mid * + * = \uparrow + \downarrow = 0 \rangle.$$

### 4.3.2 $cl(\uparrow*, \downarrow*)$

The set  $cl(\uparrow*, \downarrow*)$  consists of all sums of the positions  $\uparrow*$ ,  $\downarrow*$ , and  $*$ . We will see that the quotient group of this closure is isomorphic to that of  $cl(\uparrow, \downarrow)$ , although the outcome partitions are not the same, so that the two misère monoids are not identical.

We begin as before by determining the outcome of a general element  $k \uparrow* + \delta*$ , where  $k < 0$  means  $k \uparrow* = |k| \downarrow*$ .

**Lemma 4.3.6.** *If  $k$  is a positive integer and  $\delta \in \{0, 1\}$ , then*

$$o^-(k \uparrow* + \delta*) = \begin{cases} \mathcal{L}^-, & \text{if } k \geq 2, \text{ or } k = 1, \delta = 0; \\ \mathcal{N}^-, & \text{if } k = \delta = 1. \end{cases}$$

*Proof.* Left can move  $(k, 1)$  to  $(k - 1, 1)$  by playing an  $\uparrow*$  to zero, or to  $(k - 1, 0)$  by playing an  $\uparrow*$  to  $*$ ; she can obtain the same positions with the opposite moves from  $(k, 0)$ . Both players can move  $(k, 1)$  to  $(k, 0)$  by playing in the  $*$ . Right can move  $(k, 1)$  to  $(k - 1, 1)$  or  $(k, 0)$  to  $(k - 1, 0)$ , by playing in an  $\uparrow*$ .

If  $k \geq 2$  then Left moves to  $(k - 1, 0) \in \mathcal{L}^-$ . Right's only moves are to left-win or next-win positions, since he can decrease  $k$  by at most 1. If  $k = 1$  then Left moves to  $(0, 1) = * \in \mathcal{P}^-$ ; Right can make the same move, if  $\delta = 1$ , but otherwise his only move is to  $(0, 0) \in \mathcal{N}^-$ . Thus both players win playing first only if  $k = \delta = 1$ .  $\square$

By symmetry we obtain the outcomes for  $k < 0$ , and the outcomes for  $k = 0$  are simply the outcomes for 0 and  $*$ . To complete the monoid computation we show that every distinct pair  $(k, \delta)$  gives a distinguishable element in this closure.

**Lemma 4.3.7.** *If  $k \neq j$  are integers and  $\delta \in \{0, 1\}$ , then  $k \uparrow* + \delta* \not\equiv j \uparrow* + \delta* \pmod{\mathcal{D}}$ .*

*Proof.* It suffices to distinguish  $k \uparrow*$  and  $j \uparrow*$ , since adding  $*$  to the distinguishing game will distinguish  $k \uparrow* + *$  and  $j \uparrow* + *$ . We have  $k \uparrow* + k \downarrow* \equiv 0 \in \mathcal{N}^-$ , while  $j \uparrow* + k \downarrow* \equiv (j - k) \uparrow*$  (or  $(k - j) \downarrow*$ ), which cannot be next-win, by Lemma 4.3.6.  $\square$

**Lemma 4.3.8.** *If  $k$  and  $j$  are integers then  $k \uparrow * + * \not\equiv j \uparrow * \pmod{\mathcal{D}}$ .*

*Proof.* The positions are distinguished by  $k \downarrow *$ , since  $k \uparrow * + * + k \downarrow * \equiv * \in \mathcal{P}^-$ , while  $j \uparrow * + k \downarrow * \equiv (j - k) \uparrow *$  is not in  $\mathcal{P}^-$ , by Lemma 4.3.6.  $\square$

As for  $cl(\uparrow, \downarrow)$ , these three little lemmas tell us precisely the monoid for the closure of  $\uparrow *$  and  $\downarrow *$ ; and as for the previous closure, we find that the quotient group is  $\mathbb{Z} \times \mathbb{Z}_2$ . The outcome partition, given by Lemma 4.3.6, is not the same as the partition for previous closure, but otherwise the monoid is the isomorphic to  $\mathcal{M}_{cl(\uparrow, \downarrow)}$ :

$$\mathcal{M}_{cl(\uparrow, \downarrow)} = \langle 0, *, \uparrow *, \downarrow * \mid * + * = \uparrow * + \downarrow * = 0 \rangle.$$

### 4.3.3 $cl(E, \overline{E})$

Recall that  $E = \{0, * \mid *\}$ , and so  $\overline{E} = \{* \mid 0, *\}$ . A general element of this closure is of the form  $kE + \delta*$ , where  $k \in \mathbb{Z}$  and  $\delta \in \{0, 1\}$ . If  $k < 0$  then  $kE$  actually represents  $|k|\overline{E}$ , since  $E + \overline{E} \equiv 0 \pmod{\mathcal{D}}$ . Based on the previous two monoid computations, we might suspect that this one too has quotient group  $\mathbb{Z} \times \mathbb{Z}_2$ , and indeed the following lemmas confirm this conjecture.

**Lemma 4.3.9.** *If  $k$  is a positive integer and  $\delta \in \{0, 1\}$ , then*

$$o^-(kE + \delta*) = \begin{cases} \mathcal{L}^-, & \text{if } k \geq 2, \text{ or } k = \delta = 1; \\ \mathcal{N}^-, & \text{if } k = 1, \delta = 0. \end{cases}$$

*Proof.* If  $k \geq 2$  or if  $k = \delta = 1$  then left can win playing first by taking the option in  $E$  to zero, moving to  $(k - 1, \delta)$ , which is either left-win or (if  $k = 1$ ) previous-win. If  $k \geq 2$  then Right's options are all to  $\mathcal{L}^-$  or  $\mathcal{N}^-$ ; if  $k = 1 = \delta$  Right's move to  $E$  and his move to  $* + *$  are both next-win. So Right has no good first move in either of these cases. If  $k = 1$  and  $\delta = 0$  then both players have a winning option to  $*$ .  $\square$

The rest of the outcome partition follows by symmetry. The next step is to establish distinguishability of every unique  $(k, \delta)$ , which is accomplished in Lemmas 4.3.10 and 4.3.11.

**Lemma 4.3.10.** *If  $k \neq j$  are integers and  $\delta \in \{0, 1\}$ , then  $kE + \delta* \not\equiv jE + \delta* \pmod{\mathcal{D}}$ .*

*Proof.* As before, it suffices to distinguish  $kE$  and  $jE$ . Adding  $k\bar{E}$  gives  $kE + k\bar{E} \in \mathcal{N}^-$  while  $jE + k\bar{E} \notin \mathcal{N}^-$ , by Lemma 4.3.9, unless  $|k - j| = 1$ . If this is the case then assume, without loss of generality, that  $k = j + 1$ . The games are now distinguished by  $k\bar{E} + 2E$ :  $kE + k\bar{E} + 2E \equiv 2E \in \mathcal{L}^-$  while  $jE + k\bar{E} + 2E \equiv (k - 1)E + k\bar{E} + 2E \equiv E \in \mathcal{N}^-$ .  $\square$

**Lemma 4.3.11.** *If  $k$  and  $j$  are integers then  $kE + * \not\equiv jE \pmod{\mathcal{D}}$ .*

*Proof.* The positions are distinguished by  $k\bar{E}$ , since  $kE + * + k\bar{E} \equiv * \in \mathcal{P}^-$ , while  $jE + k\bar{E} \equiv (j - k)E$  is not in  $\mathcal{P}^-$ , by Lemma 4.3.9.  $\square$

Once again we find the quotient group of this closure of dicots to be  $\mathbb{Z} \times \mathbb{Z}_2$ . Together with the slightly different outcome partition, as given by Lemma 4.3.9, we have the misère monoid

$$\mathcal{M}_{cl(E, \bar{E})} = \langle 0, *, E, \bar{E} \mid * + * = E + \bar{E} = 0 \rangle.$$

#### 4.3.4 $cl(*_2)$

We have seen that  $*_2 = \{0, * \mid 0, *\}$  is the only day-2 dicot with no additive inverse modulo modulo  $\mathcal{D}$ . Its closure is likewise unique among these positions, in that it is impartial and finite. The following results were first observed by Plambeck in his initial study of impartial misère quotients [15].

To begin we know only that a general position is of the form  $k *_2 + \delta *$ , where  $k$  is a *positive* integer and  $\delta \in \{0, 1\}$ . The first lemma below establishes the outcome of a general position; from there we will find that although  $*_2 + *_2 \not\equiv 0$  (modulo  $\mathcal{D}$  or even modulo  $cl(*_2)$ ), it is the case that three or more copies of  $*_2$  reduce to either  $*_2$  or  $2(*_2)$ , modulo the closure of  $*_2$ .

**Lemma 4.3.12.** *If  $k$  is a positive integer and  $\delta \in \{0, 1\}$ , then*

$$o^-(k *_2 + \delta *) = \begin{cases} \mathcal{N}^-, & \text{if } \delta = 1 \text{ or if } k \equiv 1 \pmod{2}, \delta = 0; \\ \mathcal{P}^-, & \text{if } k \equiv 0 \pmod{2}, \delta = 0. \end{cases}$$

*Proof.* If  $\delta = 1$  and  $k > 0$  then any player can move either to  $k *_2 + 0 \in \mathcal{P}^-$  if  $k \equiv 0 \pmod{2}$ , by playing the  $*$  to zero, or to  $(k - 1) *_2 + * + * \equiv (k - 1) *_2 + 0 \in \mathcal{P}^-$  if

$k \equiv 1 \pmod{2}$ , by playing a  $*_2$  to  $*$ . If  $\delta = 0$  and  $k \equiv 1 \pmod{2}$  then any player can move to  $(k-1)*_2 \in \mathcal{P}^-$ .

If  $\delta = 0$  and  $k \equiv 0 \pmod{2}$  then the only moves are to play a  $*_2$  to zero, leaving  $(k-1)* \in \mathcal{N}^-$ , or to play a  $*_2$  to  $*$ , leaving  $(k-1)*+* \in \mathcal{N}^-$ . Thus the position is previous-win in this case.  $\square$

From the lemma we can already see that the precise value of  $k$  does not affect the outcome of  $k*_2+\delta*$ ; instead, only the parity of  $k$  matters. When  $k$  is odd,  $k(*_2)$  has the same outcome as any  $j(*_2)$  with  $j$  also odd; likewise for  $k$  even, with one notable exception:  $2(*_2) \in \mathcal{P}^-$  does not have the same outcome as  $0(*_2) \in \mathcal{N}^-$ . These observations are summarized in the following lemma.

**Lemma 4.3.13.** *If  $k \equiv 0 \pmod{2}$  and  $k \geq 2$ , then  $k(*_2) \equiv 2(*_2) \pmod{cl(*_2)}$ . If  $k \equiv 1 \pmod{2}$ , then  $k(*_2) \equiv *_2 \pmod{cl(*_2)}$ .*

*Proof.* Let  $X$  be any position in  $cl(*_2)$ . So  $X = j(*_2) + \delta*$  where  $\delta \in \{0, 1\}$ , and  $(k(*_2) + X) = (k+j)(*_2) + \delta*$ . By Lemma 4.3.12, the outcome of  $(k+j)(*_2) + \delta*$  is determined by the parity of  $k+j$  and the value of  $\delta$ ; but the parity of  $k+j$  is the same as the parity of  $2+j$ , if  $k$  is even, or  $1+j$ , if  $k$  is odd. Thus  $o^-(k(*_2)+X) = o^-(2(*_2)+X)$  or  $o^-(k(*_2)+X) = o^-((*_2)+X)$ , if  $k$  is even or odd, respectively, and the equivalencies follow.  $\square$

The equivalencies of Lemma 4.3.13 do require a restriction from  $\mathcal{D}$  to  $cl(*_2)$ . In the larger universe of all dicots,  $*_2$  is distinguished from  $3*_2$  by  $X = \{0 \mid \uparrow*\}$ , since  $*_2 + X \in \mathcal{N}^-$  and  $3*_2 + X \in \mathcal{L}^-$ .

At this point we might like to conclude that the closure of the game  $*_2$  has a finite quotient group of six elements:  $0, *, *_2, 2(*_2), *_2 + *, 2(*_2) + *$ . However, we must first establish that these six positions are pairwise distinct modulo  $cl(*_2)$ . The two previous-win positions,  $*$  and  $2(*_2)$ , are distinguished by  $2(*_2) + *$ , since  $*+2(*_2)+* \in \mathcal{P}^-$  while  $2(*_2)+2(*_2)+* \equiv 2(*_2)+* \in \mathcal{N}^-$ . For the next-win positions,  $*$  distinguishes  $2(*_2) + *$  from the other two nonzero games, since  $2(*_2) + * + * \in \mathcal{P} - *$  while  $*_2 + *, *_2 + * + * \in \mathcal{N}^-$ . Also,  $2(*_2) + *$  is distinguished from zero, by  $2(*_2)$ . The games  $*_2$  and  $*_2 + *$  are distinguished by adding  $*_2$ , as the former becomes a previous-win and the latter a next-win. Finally, zero is distinguished from each of these by  $*$ , since  $*_2 + * \in \mathcal{N}^-$  and  $*_2 + * + * \in \mathcal{N}^-$  while  $0 + * \in \mathcal{P}^-$ .

We can now at last give the misère monoid for  $cl(*_2)$ :

$$\mathcal{M}_{cl(*_2)} = \langle 0, *, *_2 \mid * + * = 0, \mathfrak{Z}(*_2) = *_2 \rangle,$$

$$\mathcal{N}^- = \{0, *_2, * + *_2, * + 2(*_2)\}, \mathcal{P}^- = \{*, 2(*_2)\}, \mathcal{R}^- = \emptyset, \mathcal{L}^- = \emptyset.$$

#### 4.4 Day-3 Dicots

This section makes a small first step onto the next tier of the dicot universe by finding an upper bound on the number of distinguishable day-3 dicot positions and then identifying a set of day-3 dicots that are invertible modulo  $\mathcal{D}$ . The upper bound is obtained using the inequalities of Theorem 4.3.2 to remove dominated options, while the question of invertibility is answered by applying the sufficiency condition for  $G + \overline{G} \equiv 0 \pmod{\mathcal{D}}$  given in Theorem 4.2.2.

Let  $G = \{G^{\mathbf{L}} \mid G^{\mathbf{R}}\}$  be an arbitrary day-3 dicot. Consider the possibilities for the set of left options  $G^{\mathbf{L}}$  (the same arguments hold for  $G^{\mathbf{R}}$ ). From the inequality  $\overline{E} \leq 0 \leq E \pmod{\mathcal{D}}$ , Theorem 4.3.2 (1), we can assume  $G^{\mathbf{L}}$  contains at most one of these three positions, since if two or more are present, then one dominates the others. Similarly, from inequalities (2)  $\overline{E} < *_2 < E$ , (3)  $\downarrow * < * < \uparrow *$ , (4)  $\downarrow * < \uparrow$ , and (5)  $\downarrow < \uparrow *$ , we know  $G^{\mathbf{L}}$  contains at most one of  $\overline{E}, *_2, E$ , at most one of  $\downarrow *, *, \uparrow *$ , at most one of  $\downarrow *, \uparrow$ , and at most one of  $\downarrow, \uparrow *$ .

Inequalities (1) and (2) are disjoint from the other three. We can consider  $G^{\mathbf{L}}$  as the union of two sets  $S$  and  $T$ , where the set  $S$  is one of

$$\emptyset, \{0\}, \{E\}, \{\overline{E}\}, \{*_2\}, \{0, *_2\},$$

and the set  $T$  is one of

$$\emptyset, \{*\}, \{\uparrow\}, \{\downarrow\}, \{\uparrow *\}, \{\downarrow *\}, \{*, \uparrow\}, \{*, \downarrow\}, \{*, \uparrow, \downarrow\}, \{\uparrow, \downarrow\}, \{\uparrow, \uparrow *\}, \{\downarrow, \downarrow *\}.$$

With six possibilities for  $S$  and twelve possibilities for  $T$ , there are 72 different sets  $S \cup T$  formed in this way. One of these must be eliminated as a choice for  $G^{\mathbf{L}}$ : if  $S$  and  $T$  are both empty then  $S \cup T = \emptyset$ , but neither  $G^{\mathbf{L}}$  nor  $G^{\mathbf{R}}$  can be empty, else  $G$  is not a (nonzero) dicot position. Thus, there are at most 71 different sets of left options and 71 different sets of right options, which gives 5041 possible dicots born on

day 3 *or earlier* — note that the nine positions of Figure 4.1 are also being counted here. Removing these gives an upper bound of 5032 distinguishable day-3 dicots in  $\mathcal{D}$ .

Compare this bound with the total number of possible non-distinguishable positions. If both  $G^L$  and  $G^R$  are any non-empty elements of the power set of  $\{0, *, \uparrow, \downarrow, \uparrow *, \downarrow *, E, \overline{E}, *_2\}$ , then there are  $(2^9 - 1)(2^9 - 1) = 262144$  possibilities for  $G$ . Removing the nine day-1 and day-2 positions as above gives a total of 262135 possible day-3 dicots. These calculations demonstrate the surprising value of Theorem 4.3.2: using only the unsophisticated technique of hand-tying, we have eliminated over 98% of the potential day-3 dicot positions.

As we turn now to the second goal of this section — building a set of invertible day-3 dicots — we immediately find an opportunity to make use of the above discussion. Recall, Theorem 4.2.2 says that if  $G + \overline{G} \in \mathcal{N}^-$ , and if all followers  $H$  of  $G$  satisfy  $H + \overline{H} \in \mathcal{N}^-$ , then  $G + \overline{G} \equiv 0 \pmod{\mathcal{D}}$ . The possible followers of a day-3 dicot are  $0, *$ , and the seven day-2 dicots. With the exception of  $*_2$ , we have seen that all of these satisfy  $H + \overline{H} \in \mathcal{N}^-$ . Thus, any day-3 dicot  $G$  that satisfies  $G + \overline{G} \in \mathcal{N}^-$ , without having an option to  $*_2$ , is invertible modulo  $\mathcal{D}$ .

To find all such positions, a computer program was written that generates the subset of the 5032 day-3 dicots (as described earlier using  $S$  and  $T$ ) that do not have an option to  $*_2$ . The program then iterates through this list of day-3 dicots  $G$ , checking the outcome of  $G + \overline{G}$ . Whenever the outcome is  $\mathcal{N}^-$ , we can conclude that  $G + \overline{G} \equiv 0 \pmod{\mathcal{D}}$ . In this way, 1819 invertible day-3 positions were identified. Note that this gives us neither an upper nor lower bound on the the number of invertible day-3 dicots: there are invertible dicots (with options to  $*_2$ , such as the dicot in Proposition 4.2.3) that are not counted here, and there may also be equivalencies among the identified games, which would reduce the actual number of distinguishable invertible day-3 dicots.

This completes the present analysis of general day-2 and day-3 dicots. In the last main section of the chapter, the focus turns to a specific dicot subuniverse.

#### 4.5 HACKENBUSH SPRIGS

The goal of this section is to describe the monoid for a subset of HACKENBUSH positions known as HACKENBUSH SPRIGS. This is joint work with Neil McKay and Richard Nowakowski [9]. Recall that the game of HACKENBUSH<sup>2</sup> is played on a graph with edges coloured blue, red, or green. If the graph is a path with one end rooted in the ground, then the position is called a HACKENBUSH STRING (or simply a *String*). Strings will be denoted with lowercase letters, such as  $g$  and  $h$ . A String with no green edges is called a red-blue String. A String with its rooted edge green and all others blue or red is called a HACKENBUSH SPRIG (or *Sprig*). An example of a disjunctive sum of Sprigs is shown in Figure 4.3. For non-colour diagrams, green edges are gray, blue edges are black, and red edges are white.

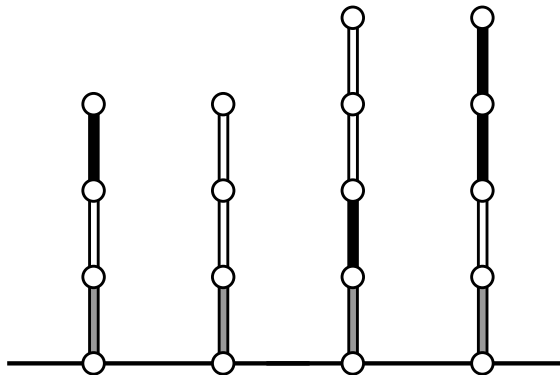


Figure 4.3: A game of HACKENBUSH SPRIGS.

Recall from normal-play game theory that the *ordinal sum* of  $G$  and  $H$  is the game  $G : H$ , with base  $G$  and dependant  $H$ . A player can move in  $H$  and leave  $G$  unaffected or can move directly to an option of  $G$  and eliminate all of  $H$ ; that is,  $G : H = \{G : H^L, G^L \mid G : H^R, G^R\}$ . A Sprig can be seen as an ordinal sum with a green edge base and a red-blue String dependant. Playing in the base brings the entire position to zero, since a single green edge is the position  $\{0 \mid 0\} = *$ . A Sprig is therefore a *star-based* ordinal sum: it is of the form  $* : g$  for a red-blue String  $g$ . Let  $\mathcal{S}$  denote the closure of of all Sprigs.

The normal-play canonical form of red-blue String is a number [4]. The lowest edge of a red-blue String  $g$  is blue if and only if  $g = \mathbf{x}$  with  $x > 0$  (that is, if  $g >^+ 0$ )

<sup>2</sup>A review of the standard rules of HACKENBUSH can be found in Appendix A.

and so the lowest edge is red if and only if  $x < 0$  ( $g <^+ 0$ ). By the ‘Colon Principle’ of [4], if  $g = \mathbf{x}$  then  $*:g = *: \mathbf{x}$ . Thus, in normal play, Sprigs are of the form  $*:\mathbf{x}$  for a number  $\mathbf{x}$ . Moreover, since  $\mathbf{x}$  is the (normal-play) canonical form of the String,  $*:\mathbf{x}$  is the (normal-play) canonical form of the Sprig [5]. The first objective of this section is to show that the misère-play canonical form of the Sprig  $*:g$  is also  $*:\mathbf{x}$ , which is done in Theorem 4.5.7. A number of preliminary results are needed. To begin, note that the misère outcome of a star-based ordinal sum, such as a Sprig, is precisely the normal-play outcome of the dependant.

**Theorem 4.5.1.** *If  $G$  is any game, then  $o^-(*:G) = o^+(G)$ .*

*Proof.* The game  $*:G$  will not end until one of the players moves in  $*$ , and at that point the player doing so loses. The only way to guarantee a win in the ordinal sum is to make the last move in  $G$ .  $\square$

As an immediate corollary we obtain the following misère outcomes for star-based numbers; recall, we are in the process of demonstrating that every Sprig is of this form. Note that, as discussed in Chapter 2, a bold  $\mathbf{x}$  represents a game identical to the normal-play canonical position with value  $x$ , while a non-bold  $x$  is used to mean the corresponding numerical value. So, for example, the inequality  $x > 0$  in the corollary below represents an inequality of actual numbers. To be consistent, the zero game is denoted  $\mathbf{0}$  in this section.

**Corollary 4.5.2.** *If  $\mathbf{x}$  is a number then*

$$o^-(*:\mathbf{x}) = \begin{cases} \mathcal{L}^-, & \text{if } x > 0; \\ \mathcal{P}^-, & \text{if } x = 0; \\ \mathcal{R}^-, & \text{if } x < 0. \end{cases}$$

As a less-immediate corollary we find that all Sprigs are invertible modulo the dicot universe. This extends the work of Sections 4.2 and 4.3, by identifying another subset of dicots that have additive inverses in  $\mathcal{D}$ . Note that the conjugate of an ordinal sum is the ordinal sum of the conjugates (by induction):

$$\overline{G:H} = \{\overline{G:H^R}, \overline{G^R} \mid \overline{G:H^L}, \overline{G^L}\} = \{\overline{G}: \overline{H^L}, \overline{G^L} \mid \overline{G}: \overline{H^R}, \overline{G^R}\} = \overline{G}: \overline{H}.$$

**Corollary 4.5.3.** *If  $g$  is a red-blue String then  $*:g + \overline{*:g} \equiv \mathbf{0} \pmod{\mathcal{D}}$ .*

*Proof.* By Theorem 4.2.2, to show  $*:g + \overline{*:g} \equiv \mathbf{0} \pmod{\mathcal{D}}$ , it suffices to show that  $*:g + \overline{*:g} \in \mathcal{N}^-$  and that the same is true for any follower of  $*:g$ . But any non-zero follower of a Sprig is also a Sprig, and since  $\overline{*:g} = *:\overline{g}$ , we need only show  $*:g + *:\overline{g} \in \mathcal{N}^-$  for any String  $g$ .

If  $g = \mathbf{0}$  then this is just Corollary 4.2.4. Otherwise,  $g$  is either left-win or right-win in normal-play, so assume without loss of generality that  $g \in \mathcal{L}^+$ . Left playing first on  $*:g + *:\overline{g}$  moves  $*:\overline{g}$  to  $\mathbf{0}$  (that is, removes the green edge) and then wins playing second under misère play on  $*:g$ , by Theorem 4.5.1, since  $g \in \mathcal{L}^+$ . By the same argument, since  $\overline{g} \in \mathcal{R}^+$ , Right can similarly win this sum playing first, and so  $*:g + *:\overline{g} \in \mathcal{N}^-$ .  $\square$

The next result gives an inequality that is true for the ordinal sum of any two games, provided the base is not an end. Moreover, the inequality holds in the universe of all misère games, not just in  $\mathcal{D}$ .

**Theorem 4.5.4.** *If  $G$  is not an end and  $H \in \mathcal{L}^+$ , then  $G:H \geq G$ .*

*Proof.* Let  $X$  be any game. Given a winning strategy in  $G + X$ , Left can do at least as well in  $G:H + X$  by following this amended strategy: if Right plays in  $H$  then respond in  $H$ , otherwise follow the original strategy for  $G + X$ . As  $H \in \mathcal{L}^+$ , Left can always respond to Right's moves in  $H$ . Since  $G$  has options for both players, the addition of  $H$  is of no benefit to Right.  $\square$

Applying Theorem 4.5.4 to the present situation, we have the following corollary. This will allow us (in Theorem 4.5.6) to compare two Sprigs in misère-play based on the normal-play comparison of the red-blue String. Again this inequality holds in the non-restricted misère universe. Note that associativity of ordinal sums is demonstrated in [4], and so  $(*:g):h = *(g:h)$ .

**Corollary 4.5.5.** *If  $g$  and  $h$  are red-blue Strings and  $h \in \mathcal{L}^+$ , then*

$$(*:g):h > *:g.$$

*Proof.* By Theorem 4.5.4,  $(*:g):h \geq *:g$ . It remains to find a game that can distinguish  $(*:g):h$  and  $*:g$ . Consider  $*:\overline{g} + *$ . The sum  $*:g + *:\overline{g} + *$  is in  $\mathcal{P}^-$ ,

as  $*:g + *:\bar{g} + * \equiv * \pmod{\mathcal{D}}$  by Corollary 4.5.3. Since the String  $h$  is left-win in normal play, Left must get the last move (that is, its bottom edge must be blue); going first in  $(*:g):h + *:\bar{g} + *$ , Left wins by moving in the bottom edge of  $h$ , to  $*:g + *:\bar{g} + * \in \mathcal{P}^-$ . Left has a good first move and so  $(*:g):h + *:\bar{g} + *$  is not in  $\mathcal{P}^-$ .  $\square$

In the following theorem we use the symbol  $>^+$  to indicate inequality in normal-play. Since  $g$  and  $h$  are numbers in normal play,  $g >^+ h$  means  $x > y$  where  $\mathbf{x}$  and  $\mathbf{y}$  are the normal-play canonical forms of  $g$  and  $h$ , respectively.

**Theorem 4.5.6.** *If  $g$  and  $h$  are red-blue Strings and  $g >^+ h$  then*

$$*:g > *:h.$$

*Proof.* Since a red-blue string is a number in normal-play,  $g$  is either  $\mathbf{0}$  or is in  $\mathcal{L}^+$ . Let  $k$  be the longest String such that  $g = k:g'$  and  $h = k:h'$  for Strings  $g'$  and  $h'$  (possibly  $k$  is zero); that is,  $k$  is the ‘common part’ of the Strings  $g$  and  $h$ , from the bottom up. At most one of  $g'$  and  $h'$  is empty, since otherwise  $g = h$ , contradicting  $g >^+ h$ . Furthermore,  $g'$  must start with a blue edge, since  $g \in \mathcal{L}^+$  means  $g' \in \mathcal{L}^+$ . Then  $h'$  must start with a red edge, else  $k$  is not maximal, and so  $h' \in \mathcal{R}^+$ . By Corollary 4.5.5,  $*:g = *(k:g') = (*:k):g' > *:k$ , because  $g' \in \mathcal{L}^+$ , and in turn  $*:k > (*:k):h'$ , because  $h' \in \mathcal{R}^+$ . Thus  $*:g > (*:k):h' \equiv *:h$ .  $\square$

Theorem 4.5.6 is critical for our analysis of misère Sprigs. It shows that the (non-zero) left and right options of a Sprig are totally ordered in general misère play: among such options, Left should choose to play to  $*:g^L$  where  $g^L$  is the best normal-play left option of  $g$ . Since we know the normal-play canonical form of  $g$  is a number  $\mathbf{x}$ , this means Left can consider  $*:g$  as  $*:\mathbf{x}$  in misère play. What remains is to show that in fact  $*:\mathbf{x}$  is the misère-play canonical form of the Sprig.

**Theorem 4.5.7.** *If  $\mathbf{x}$  is the normal-play canonical form of the red-blue String  $g$ , then  $*:\mathbf{x}$  is the misère-play canonical form of  $*:g$ .*

*Proof.* Let  $\mathbf{x} = \{\mathbf{x}^L \mid \mathbf{x}^R\}$  be the number that is the normal-play canonical form of the red-blue String  $g$ . According to [4], in normal-play, if Left moves in  $g$  then she should move to  $*:\mathbf{x}^L$ , and if Right moves in the red-blue part he should move to

$*: \mathbf{x}^R$ . By Theorem 4.5.6, the same moves dominate in misère play; thus, so far, we have  $*: g = \{\mathbf{0}, *: \mathbf{x}^L \mid \mathbf{0}, *: \mathbf{x}^R\}$ , where Left can move to zero by chopping the green edge or to  $\mathbf{x}^R$  by making the best normal-play move in the red-blue part, and similarly for Right. The two left (right) options are incomparable, since  $* + \mathbf{0} \in \mathcal{P}^-$  while  $* + *: H \in \mathcal{N}^-$  for any  $H$ , as the first player can eliminate  $*: H$  and leave  $*$ .

It remains to show that there are no reversible options to bypass; that is, that there is no left option  $*: \mathbf{x}^L$  to which Right has a response ( $\mathbf{0}$  or  $*: \mathbf{x}^{LR}$ ) that is less than the original game  $S$ , and likewise, that there is no reversible Right option. By symmetry it suffices to show the former. We already know  $\mathbf{0}$  is incomparable to a Sprig, so it suffices to show  $*: \mathbf{x}^{LR} \geq \mathbf{x}$ . By Proposition 2.3.2,  $x^{LR} > x$ , and so by Theorem 4.5.6,  $*: \mathbf{x}^{LR} \not\geq *: \mathbf{x}$ . Thus, there are indeed no reversible options in  $*: g$ , and so the misère-play canonical form of the Sprig is  $\{0, *: \mathbf{x}^L \mid 0, *: \mathbf{x}^R\} = *: \mathbf{x}$ .  $\square$

With the canonical form of a HACKENBUSH SPRIG established, we can work towards determining the outcome of a general sum of Sprigs. By Theorem 4.5.7, such a sum is of the form

$$\sum_{x \in X} *: \mathbf{x} + \sum_{y \in Y} *: \bar{\mathbf{y}} + \delta *,$$

where  $X$  and  $Y$  are multisets of positive numbers, and  $\delta$  is 0 or 1. We can represent part of this position by the ordered pair  $(X, Y)$ , and then an arbitrary sum of Sprigs is either  $(X, Y)$  or  $(X, Y) + *$ . Sprigs which Left wins — that is, Sprigs  $*: \mathbf{x}$  with  $x > 0$  — are referred to as ‘positive’ or ‘Left’ Sprigs, and similarly for Right.

By Corollary 4.5.3, we know that if there are any numbers in common between the sets  $X$  and  $Y$ , then the terms  $*: \mathbf{x}$  and  $*: \bar{\mathbf{x}}$  cancel in the sum. Let  $G = (X, Y)$  be called *reduced* if all such cancellations have been made; that is, if  $X \cap Y = \emptyset$ . Given a position  $G = (X, Y)$ , there is a unique reduced position  $G' = (X', Y') \equiv G \pmod{\mathcal{D}}$ , where  $X' = X \setminus Y$  and  $Y' = Y \setminus X$ .

Recall from the analysis of alternating games in Chapter 3, if Left has many more games ‘in her favour’ than Right, then a sum is left-win, and vice versa for Right, and only when the so-called ‘tilt’ is hovering around zero do other factors come into play. Here a similar observation is made: if there are many more positive Sprigs than negative Sprigs, then the sum is left-win; however, if the difference between  $|X|$  and  $|Y|$  is small, then the actual values of the numbers in  $X$  and  $Y$  affect the overall

outcome. Define the *advantage* of  $G = (X, Y)$  to be  $\Delta(G) = |X| - |Y|$  (which is the same as  $|X'| - |Y'|$ ), and the *lean* of  $G$  to be  $\epsilon(G) = \min(X') - \min(Y')$ . If  $X'$  or  $Y'$  is empty then set  $\epsilon(G) = 0$ . Note that the lean is the difference of the minimum values in the *reduced* sum, after cancellation, and so the lean cannot be zero unless one of  $X', Y'$  is empty. The concepts of advantage and lean allow us to describe the outcome of any game  $(X, Y)$  or  $(X, Y) + *$ .

First note that if both the advantage and lean are zero, then the sum (after cancellation) is also zero.

**Lemma 4.5.8.** *If  $G = (X, Y)$  with  $\Delta(G) = 0$  and  $\epsilon(G) = 0$ , then  $X = Y$  and  $G \equiv 0 \pmod{\mathcal{D}}$ .*

*Proof.* As  $\Delta(G) = 0$ ,  $|X| = |Y|$  and  $|X'| = |Y'|$ . As  $\epsilon(G) = 0$ , at least one of  $X'$  or  $Y'$  must be empty, so they both must be, as they are the same size.  $\square$

Next we see that if a sum of Sprigs has no single  $*$  then the outcome is completely determined by the advantage — that is, by the difference in the number of positive Sprigs and the number of negative Sprigs.

**Theorem 4.5.9.** *If  $G = (X, Y)$  then*

$$o^-(G) = \begin{cases} \mathcal{L}^- & \text{if } \Delta(G) > 0; \\ \mathcal{R}^- & \text{if } \Delta(G) < 0; \\ \mathcal{N}^- & \text{if } \Delta(G) = 0. \end{cases}$$

*Proof.* Let  $G = (X, Y)$ . We proceed by induction on  $|X| + |Y|$ . If  $|X| + |Y| = 0$  then  $G = 0$ , which is a next-player win. If  $|X| = 1$  and  $|Y| = 0$  then  $G \in \mathcal{L}^-$  by Theorem 4.5.1.

Suppose  $|X| = |Y| > 0$ . As  $|Y| > 0$ , Left going first can move some  $*$ :  $\bar{\mathbf{y}}$  to 0, leaving  $G^L$  with  $\Delta(G^L) > 0$ , which is left-win by induction. Similarly, Right can win moving first. Thus, if  $\Delta(G) = 0$ , then  $G \in \mathcal{N}^-$ .

Suppose  $|X| > |Y|$ . If  $|Y| > 0$ , then Left wins going first as above. If  $|Y| = 0$  then, given the base case, we need only consider  $|X| > 1$ . Left wins going first by moving  $*$ :  $\mathbf{x}$  to  $\mathbf{0}$  for some  $x \in X$ , which is a winning move by induction. If Right moves first to  $G^R$  such that  $\Delta(G^R) \geq 0$ , then Left wins because  $G^R \in \mathcal{N}^- \cup \mathcal{L}^-$ ;

if Right's move does not change the advantage, then Left wins using the first-player argument previously given. This shows the first two cases; by symmetry,  $G \in \mathcal{R}^-$  if  $|X| < |Y|$ .  $\square$

Thus, when playing a sum of Sprigs with no single green edge (or, technically, with an even number of them, as they will then cancel to zero), the outcome is determined by the advantage alone. The addition of a single green edge,  $*$ , complicates things slightly. The following lemma prepares us for Theorem 4.5.11, which shows that the outcome of  $(X, Y) + *$  is determined by both the advantage and the lean of  $(X, Y)$ .

**Lemma 4.5.10.** *If  $G = (X, Y)$  with  $\Delta(X) > 0$ , then Left can win  $G + *$  playing first, and if Right can win playing first, he can do so moving  $*: \mathbf{x}$  to  $\mathbf{0}$ , for  $x = \max(X')$ .*

*Proof.* Playing in  $G + *$ , if  $\Delta(G) > 0$  then Left can win by playing first by moving  $*$  to  $\mathbf{0}$ , leaving  $G$  which is winning by Theorem 4.5.9.

If Right does not move a Sprig to  $\mathbf{0}$ , then he must play in some  $*: \mathbf{x}$  to  $*: \mathbf{x}^R$ . As  $x^R > x$ , Theorem 4.5.6 says that  $*: \mathbf{x}^R > *: \mathbf{x}$ . Thus this right option is better for Left than the original game, and so Left wins playing first from here. We conclude that Right can only win by eliminating a Sprig. By Theorem 4.5.6, Right's options in this case are totally ordered, and so Right should eliminate the Sprig that is best for the opponent; that is, the Sprig  $*: \mathbf{x}$  with largest value of  $x$ .  $\square$

We can now describe the outcome of a sum of Sprigs with an odd number of  $*$  positions (that is, after cancellation, sums with a single  $*$  position).

**Theorem 4.5.11.** *If  $G = (X, Y)$  then*

$$o^-(G + *) = \begin{cases} \mathcal{L}^- & \text{if } \Delta(G) > 1, \text{ or } \Delta(G) = 0, 1 \text{ and } \epsilon(G) > 0; \\ \mathcal{R}^- & \text{if } \Delta(G) < -1, \text{ or } \Delta(G) = 0, -1 \text{ and } \epsilon(G) < 0; \\ \mathcal{N}^- & \text{if } \Delta(G) = 1 \text{ and } \epsilon(G) \leq 0, \text{ or } \Delta(G) = -1 \text{ and } \epsilon(G) \geq 0; \\ \mathcal{P}^- & \text{if } \Delta(G) = 0 \text{ and } \epsilon(G) = 0. \end{cases}$$

*Proof.* The outcome is the same as the outcome of the reduced game, so assume  $G = (X, Y)$  is reduced and let  $H = G + *$ . We proceed by induction on  $|X| + |Y|$ .

If  $|X| + |Y| = 0$  then  $H = *$ , which is previous-win. If  $|X| + |Y| = 1$  then  $H \in \mathcal{N}^-$ , as either player wins by moving to  $*$ . If  $|X| = |Y| = 1$ , then  $H = *: \mathbf{x} + *: \bar{\mathbf{y}} + *$ .

The first player to move any of the three Sprigs to  $\mathbf{0}$  loses; play proceeds in  $\mathbf{x}$  and  $\overline{\mathbf{y}}$  until someone is forced to do so, at which point the opponent responds by eliminating either  $*$  or the ‘opponent’s Sprig’ (that is, Left eliminates  $*: \overline{\mathbf{y}}$  and Right eliminates  $*: \mathbf{x}$ ). Left then wins on either  $*$  or  $*: \mathbf{x}$ , or Right wins on  $*$  or  $*: \overline{\mathbf{y}}$ .

If  $|X| = |Y| > 1$  (so  $\Delta(G) = 0$ ) and  $\epsilon(G) > 0$ , then Left wins going first in  $H$  by moving one of Right’s Sprigs to 0, changing her advantage to 1 and thereby leaving a position with advantage 1 and positive lean, which is Left-win by induction. We need to show that Left also wins playing second. Right’s options are to eliminate the  $*$ , leaving a next-win position by Theorem 4.5.9, or to eliminate one of Left’s Sprigs, leaving a position with advantage  $-1$  and non-negative lean, which is next-win by induction. Thus, Right has no good first move and this position is left-win. The case  $\epsilon(G) < 0$  follows by symmetry.

If  $|X| > |Y|$  then Left can win going first by Lemma 4.5.10. Right’s best move going first is to move a Sprig to 0. If  $|X| - |Y| > 2$ , Right’s move loses, as the position is still left-win by induction. However, if  $|X| - |Y| = 1$  and  $\epsilon(G) < 0$ , then this Right move is winning, as the resulting position has zero advantage and negative lean, which is right-win by induction. The case  $|X| < |Y|$  follows by symmetry.  $\square$

This completes the outcome partition portion of the monoid computation. Lastly we must determine the distinguishability of the Sprigs. In fact, every unique *reduced* pair of multisets  $(X, Y)$  gives distinguishable games  $(X, Y)$  and  $(X, Y) + *$ . This is established in the following two lemmas. From here the results hold only modulo the closure of Sprigs,  $\mathcal{S}$ .

**Lemma 4.5.12.** *Let  $G = (X_1, Y_1)$  and  $H = (X_2, Y_2)$  be reduced games. If  $\delta \in \{0, 1\}$ , then  $G + \delta \cdot * \equiv H + \delta \cdot * \pmod{\mathcal{S}}$  if and only if  $X_1 = X_2$  and  $Y_1 = Y_2$ .*

*Proof.* The sufficiency is clear. Conversely, suppose without loss of generality that  $X_1 \neq X_2$ . In this case, the reduced form of  $G + \overline{H}$  is not 0. The games  $G + *$  and  $H + *$  are now distinguished by  $\overline{H}$ , since by Lemma 4.5.11,  $G + \overline{H} + * \notin \mathcal{P}^-$  while  $H + \overline{H} + * \equiv * \pmod{\mathcal{D}}$  and  $* \in \mathcal{P}^-$ . Similarly,  $G$  and  $H$  are distinguished by  $\overline{H} + *$ .  $\square$

**Lemma 4.5.13.** *If  $G = (X_1, Y_1)$  and  $H = (X_2, Y_2)$ , then  $G \not\equiv H + * \pmod{\mathcal{S}}$ .*

*Proof.* The games are distinguished by  $\overline{H}$ , since  $H + \overline{H} + * \in \mathcal{P}^-$  while  $G + \overline{H} \notin \mathcal{P}^-$ .  $\square$

With the reductions  $*+* \equiv 0 \pmod{\mathcal{D}}$  and  $*:\mathbf{x}+*:\overline{\mathbf{x}} \equiv \mathbf{0} \pmod{\mathcal{D}}$ , we can now describe the misère monoid associated with the closure of Sprigs. By the invertibility of Sprigs,  $\mathcal{M}_S$  is actually a group. It is generated by the set  $\mathbb{Q}_2$  of dyadic rationals. If we allow  $\mathbf{x}$  to be zero, then the Sprig  $*$  is also of the form  $*:\mathbf{x}$  for a dyadic rational  $\mathbf{x}$ . Let  $\mathbb{Q}_2^+$  denote the set of all positive dyadic rationals.

$$\mathcal{M}_S = \langle 0, *:\mathbf{x}, *:\overline{\mathbf{x}} \mid \mathbf{x} \in \mathbb{Q}_2^+ \cup \{\mathbf{0}\}, *:\mathbf{x} + *:\overline{\mathbf{x}} = 0 \rangle,$$

with outcome tetra-partition given implicitly by Theorems 4.5.9 and 4.5.11.

## 4.6 Future Directions for Dicot Games

Many of the findings of this chapter have the potential to be extended or generalized. Given the ease with which the monoids of Section 4.3 were computed, it is reasonable to expect that the monoids for larger groups of day-2 dicots can be described: for example,  $cl(\uparrow, \downarrow, \uparrow*, \downarrow*, E, \overline{E})$ , or the closure of a subset of these games, may follow without much difficulty from the work already completed.

The results of Section 4.4 can likewise be continued, to find other instances of domination among day-3 dicots, and to also consider reversibility. In this way it may be possible to determine the unique day-3 dicots, by finding the canonical form of each.

Given the invertibility criteria of Theorem 4.2.2, it would be interesting to consider other specific games, such as HACKENBUSH SPRIGS, to see if this and other results can lead to a complete solution. A natural way to generalize Sprigs is to consider other ordinal sums  $G:\mathbf{x}$  where the base  $G$  is a dicot and the dependant  $\mathbf{x}$  is a normal-play number. It is possible that much of the analysis of Section 4.5 would apply to other such games.

## Chapter 5

### The Dead-ending Universe

#### 5.1 Introduction to Dead-ending Games

In many games, players take turns placing pieces on a board according to some set of rules. Usually these rules imply that the board spaces available to a player on his or her turn are a subset of those available on the previous turn; the games DOMINEERING, COL, SNORT, HEX, and NOGO, among many others, fit this description<sup>1</sup>. Among the researchers very recently gaining interest in such games, the term *placement games* has been introduced. In contrast to games like MAZE or KONANE, placement games have the property that a player cannot ‘open up’ moves for him or herself, or for the opponent; in particular, if a player has no available moves at some position of the game then they will have no moves in any follower of that position. This particular property, called *dead-ending*, forms the third and final universe studied in this thesis.

Recall from Definition 2.1.8 that a *left end* is a position with no options for Left, and a *right end* is a position with no options for Right. The zero game can be considered both a left end and a right end. The terms ‘end position’ or ‘end game’ are used to mean a position that is either a left end or a right end, the term ‘end follower’ means a follower that is a left or right end, and so on. *Dead-ending* games are defined as follows.

**Definition 5.1.1.** *A left (right) end is dead if every follower is also a left (right) end. A game  $G$  is called dead-ending if every end follower of  $G$  is dead.*

Figure 5.1 illustrates the definition of dead-ending games. The top three positions do not satisfy the dead-ending property: the first is a right end, but is not dead, as Right can move in one of its followers; the second is similarly a left end that is not dead, and the third is a non-end that is not dead-ending. In each case either the original position or one of its followers is an end that is still ‘alive’ for both players.

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<sup>1</sup>Rules for each of the games mentioned in this section are included in Appendix A.

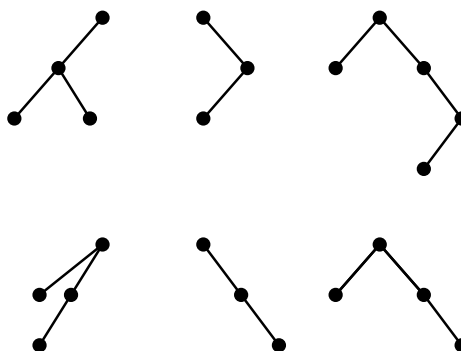


Figure 5.1: Three games that are not dead-ending (top) and three that are (bottom).

The bottom three games are dead-ending. The first is a dead right end: Right has no move now, and has none in any follower. The second is a dead left end, and the third is a (non-end) dead-ending game because all of its followers are dead ends.

Let the set of all dead-ending games be denoted  $\mathcal{E}$ . In addition to the games listed above, many well-studied positions from normal-play game theory have the dead-ending property: integers are dead ends, and non-integer numbers, all-small games, and all HACKENBUSH positions are dead-ending. The set of all dead-ending games is thus a meaningful (and large) universe to consider; moreover, it is a natural extension of the dicot universe, since every dicot game is also dead-ending. The purpose of this chapter is to explore several significant subuniverses of  $\mathcal{E}$ . Along the way the misère monoid for all normal-play canonical form numbers is discovered for the first time. The first four sections are based on joint work with Gabriel Renault [12].

Section 5.2 begins with some immediate consequences of the definition of dead-ending. Section 5.3 presents an analysis of ends in the dead-ending universe, which includes all integers in normal-play canonical form. In Section 5.4 this analysis is extended to non-integer numbers and we find that the monoid of all numbers is equivalent to the monoid of integers. The partial orders of these subuniverses (modulo the subuniverse as well as modulo  $\mathcal{E}$ ) are also determined, as is the invertibility of the elements (modulo  $\mathcal{E}$ ). Section 5.5 is a short discussion of other dead-ending games, specifically in the context of equivalency to zero modulo the dead-ending universe. Finally, Section 5.6 presents a complete solution to the dead-ending game PARTIZAN KAYLES (which can be seen as a variant of DOMINEERING), and Section 5.7 talks about the promising future of both dead-ending and placement games.

## 5.2 Preliminary Results for Dead-ending Games

A first look around the newly defined universe  $\mathcal{E}$  reveals a number of pleasant properties. To begin, Lemmas 5.2.1 and 5.2.2 show that the set of dead-ending games is ‘closed’ in two important respects: it is closed under followers and closed under disjunctive sum.

**Lemma 5.2.1.** *If  $G$  is dead-ending then every follower of  $G$  is dead-ending.*

*Proof.* If  $H$  is a follower of  $G$ , then every follower of  $H$  is also a follower of  $G$ ; thus if  $G$  satisfies the definition of dead-ending, then so does  $H$ .  $\square$

**Lemma 5.2.2.** *If  $G$  and  $H$  are dead-ending then  $G + H$  is dead-ending.*

*Proof.* Any follower of  $G + H$  is of the form  $G' + H'$  where  $G'$  and  $H'$  are (not necessarily proper) followers of  $G$  and  $H$ , respectively. If  $G' + H'$  is a left end, then both  $G'$  and  $H'$  are left ends, which must be dead, since  $G$  and  $H$  are dead-ending. Thus, any right options  $G'^R$  and  $H'^R$  are left ends, and so all options  $G'^R + H'$  and  $G' + H'^R$  of  $G' + H'$  are left ends. A symmetric argument holds if  $G' + H'$  is a right end, and so  $G + H$  is dead-ending.  $\square$

Thus, like dicot games, and unlike alternating games, dead-ending games are closed under addition. This gives additional weight to the claim that  $\mathcal{E}$  is a very natural subuniverse of misère games.

The next lemma establishes the outcome of dead left and right ends. Note that Left trivially wins any left end playing first under misère play. In general, Left may or may not win a left end playing second; for example, the game  $\{\cdot \mid \mathbf{1}\}$  is a left end in  $\mathcal{N}^-$ . If a (non-zero) left end is dead, however, then it is a win for Left playing first or second.

**Lemma 5.2.3.** *If  $G \neq 0$  is a dead left end then  $G \in \mathcal{L}^-$ , and if  $G \neq 0$  is a dead right end then  $G \in \mathcal{R}^-$ .*

*Proof.* A left end is always in  $\mathcal{L}^-$  or  $\mathcal{N}^-$ . If  $G$  is a dead left end then any right option  $G^R$  is also a left end, so Right has no good first move. Similarly, a dead right end is in  $\mathcal{R}^-$ .  $\square$

In the next section we will find that in a sum of dead ends, the best strategy is for a player to bring his or her ends to zero as quickly as possible. Even for non-end dead-ending positions, this concept of ‘the shortest route to zero’ has great influence on the outcome of sums. Thus, the following functions are defined.

**Definition 5.2.4.** *Let  $G$  be a game with a non-alternating path to zero for Left. The left-length of  $G$ , denoted  $l(G)$ , is the minimum number of consecutive left moves required for Left to reach zero in  $G$ . Similarly, if  $G$  has a non-alternating path to zero for Right, then the right-length  $r(G)$  of  $G$  is defined as the minimum number of consecutive right moves required for Right to reach zero in  $G$ .*

In general, left-length is well-defined if, in addition to  $G$  having a non-alternating path to zero for Left, the shortest of such paths is never dominated by another option. The latter condition ensures  $l(G) = l(G')$  when  $G \equiv G'$ . The symmetric requirement for Right ensures  $r(G)$  is well-defined. These conditions are met if  $G$  is a (normal-play) canonical-form number or if  $G$  is an end in  $\mathcal{E}$ . If  $l(G)$  and  $l(H)$  are both well-defined then  $l(G + H)$  is defined and  $l(G + H) = l(G) + l(H)$ . Similarly, when right-length is defined for  $G$  and  $H$ , we have  $r(G + H) = r(G) + r(H)$ .

The functions left-length and right-length are critical for analyzing dead ends, the subject of Section 5.3. They also affect the outcome of other dead-ending games, such as the non-integer numbers discussed in Section 5.4.

### 5.3 Integers and Other Dead Ends

Recall that  $\mathbf{n}$  denotes the game  $\{\mathbf{n} - \mathbf{1} \mid \cdot\}$ , where  $\mathbf{0} = 0 = \{\cdot \mid \cdot\}$ . That is,  $\mathbf{n}$  is identical to the normal-play canonical form of the integer  $n$ . To be consistent with notation, for the remainder of this chapter the game  $\{\cdot \mid \cdot\}$  will also be denoted by a bold  $\mathbf{0}$ . Integer positions that are ‘negative’ in normal play are generally denoted here in the usual misère way:  $\overline{\mathbf{n}} = \{\cdot \mid \overline{\mathbf{n} - \mathbf{1}}\}$ . However,  $-\mathbf{n}$  may be used for the same purpose, when it is more readable to do so, in cases where invertibility has been established.

In normal play, games with integer values interact with each other as numerical integers do, with respect to addition and ordering in particular. As discussed in Chapter 2, the same can not be said for misère play. One property that does hold

in both normal and misère play is that the disjunctive sum of positive integers  $\mathbf{n}$  and  $\mathbf{m}$  is the integer  $\mathbf{n} + \mathbf{m}$ , although this is not generally true (in misère games) if one of  $n$  or  $m$  is negative and the other positive. This section proves that the restricted universe of integers under misère play has much of the structure enjoyed by normal-play integers.

An integer is an example of a dead end: if  $n > 0$  then Right has no move in  $\mathbf{n}$  and no move in any follower of  $\mathbf{n}$ . Similarly, if  $n < 0$  then  $\mathbf{n}$  is a dead left end. Thus, the following results for ends in the dead-ending universe are true for all integers, modulo  $\mathcal{E}$ . Since the sum of a dead left end and a dead right end may not be a dead end (or any end at all), the set of dead ends is not closed under disjunctive sum; thus, when restricting our focus to dead ends, the universe we consider is actually the *closure* of dead ends, or in other words the set of all sums of dead ends. To be consistent with the notation for alternating ends, denote this universe by  $\mathcal{E}_e$ .

An immediate result for this subuniverse (and indeed any larger universe) is given below. Essentially, when all games in a sum are dead ends, the outcome is completely determined by the left- and right-lengths of the games.

**Lemma 5.3.1.** *If  $G$  is a dead right end and  $H$  is a dead left end, then*

$$o^-(G + H) = \begin{cases} \mathcal{N}^- & \text{if } l(G) = r(H), \\ \mathcal{L}^- & \text{if } l(G) < r(H), \\ \mathcal{R}^- & \text{if } l(G) > r(H). \end{cases}$$

*Proof.* Each player has no choice but to play in their own game, and so the winner will be the player who can run out of moves first.  $\square$

Lemma 5.3.1 is used to prove the following theorem, which demonstrates the invertibility of all ends in  $\mathcal{E}$ . In particular, this shows that every integer has an additive inverse modulo  $\mathcal{E}$ .

**Theorem 5.3.2.** *If  $G$  is a dead end then  $G + \overline{G} \equiv \mathbf{0} \pmod{\mathcal{E}}$ .*

*Proof.* Assume without loss of generality that  $G \neq \mathbf{0}$  is a dead right end. Since every follower of a dead end is also a dead end, Theorem 2.4.2 applies, with  $S$  the set of all dead left and right ends. It therefore suffices to show  $G + \overline{G} + X \in \mathcal{L}^- \cup \mathcal{N}^-$  for any left

end  $X$  in  $\mathcal{E}$ . We have  $l(G) = r(\overline{G})$  and  $r(X) \geq 0$ , so  $l(G) \leq r(\overline{G}) + r(X) = r(\overline{G} + X)$ , which gives  $G + \overline{G} + X \in L^- \cup \mathcal{N}^-$  by Lemma 5.3.1.  $\square$

**Corollary 5.3.3.** *If  $n$  is an integer then  $\mathbf{n} + \overline{\mathbf{n}} \equiv \mathbf{0}$  (mod  $\mathcal{E}$ ).*

Note that equivalency in  $\mathcal{E}$  implies equivalency in all subuniverses of  $\mathcal{E}$ ; thus in the universe of integers alone, every game has an inverse.

Lemma 5.3.1 shows that when playing a sum of dead ends, both players aim to exhaust their own options as quickly as possible. This suggests that options with longer paths to zero will be dominated by shorter paths; in particular, we have that integers are totally ordered among dead ends, as established in Theorem 5.3.4 below. Note that this ordering only holds in the subuniverse of the closure of dead ends, and not in the whole universe  $\mathcal{E}$ . In fact, we see immediately in Theorem 5.3.5 that distinct integers are pairwise incomparable modulo  $\mathcal{E}$ , just as they are in the general misère universe.

The following arguments frequently use the fact that, when  $H \in \mathcal{U}$  has an additive inverse modulo  $\mathcal{U}$ ,  $G \succeq H$  (mod  $\mathcal{U}$ ) if and only if  $G + \overline{H} \succeq \mathbf{0}$ . Recall that  $\mathcal{E}_e$  denotes the closure of dead ends.

**Theorem 5.3.4.** *If  $n < m \in \mathbb{Z}$  then  $\mathbf{n} \not\succeq \mathbf{m}$  (mod  $\mathcal{E}_e$ ).*

*Proof.* Note that  $\mathbf{n} \succeq \mathbf{m}$  if and only if  $\mathbf{n} + \overline{\mathbf{m}} \succeq \mathbf{m} + \overline{\mathbf{m}}$ , which, by Corollary 5.3.3, is if and only if  $\mathbf{n} + \overline{\mathbf{m}} \succeq \mathbf{0}$ . Note also that by the same corollary we have  $\mathbf{n} + \overline{\mathbf{m}}$  is equal to the game  $\mathbf{k}$ , where  $k = n - m$ . Since  $n < m$ , we have  $k < 0$ , and so  $\mathbf{k}$  is a negative integer.

Thus, it suffices for  $\mathbf{n} + \overline{\mathbf{m}} \succeq \mathbf{0}$  (mod  $\mathcal{E}_e$ ) to show that  $\mathbf{k} \succeq \mathbf{0}$  (mod  $\mathcal{E}_e$ ), for any negative integer  $k$ . Let  $X$  be any game in the closure  $\mathcal{E}_e$  of dead ends; then  $X = Y + Z$  where  $Y$  is a dead right end and  $Z$  is a dead left end. Suppose Left wins  $X$  playing first; then by Lemma 5.3.1,  $l(Y) \leq r(Z)$ . We need to show Left wins  $\mathbf{k} + X$ , so that  $o^-(\mathbf{k} + X) \geq o^-(X)$ . Since  $k$  is a negative integer,  $r(\mathbf{k})$  is defined and  $r(\mathbf{k}) = -k > 0$ . Thus  $l(Y) \leq r(Z) < r(Z) + r(\mathbf{k}) = r(Z + \mathbf{k})$ , which gives  $\mathbf{k} + Y + Z = \mathbf{k} + X \in \mathcal{L}^- \cup \mathcal{N}^-$ , by Lemma 5.3.1.  $\square$

In general,  $G \geq H$  under misère play implies  $G \geq H$  under normal play [19]; Theorem 8 shows this is not always the case for misère inequality modulo a restricted universe.

**Theorem 5.3.5.** *If  $n \neq m \in \mathbb{Z}$  then  $\mathbf{n} \parallel \mathbf{m} \pmod{\mathcal{E}}$ .*

*Proof.* Assume  $n > m$ . Then we have  $\mathbf{n} \not\leq \mathbf{m} \pmod{\mathcal{E}}$ , because  $\mathbf{n} + \overline{\mathbf{m}} \in \mathcal{R}^-$  while  $\mathbf{m} + \overline{\mathbf{m}} \equiv \mathbf{0} \in \mathcal{N}^-$ . It remains to show  $\mathbf{n} \not\geq \mathbf{m}$ .

Define a family of games  $\lambda_k$  by

$$\lambda_1 = \{\mathbf{0} \mid \overline{\mathbf{1}}\}, \lambda_k = \{\mathbf{0} \mid \lambda_{k-1}\}.$$

Note that  $\mathbf{n} + \lambda_n \in \mathcal{L}^-$ , since Left wins playing first or second by ignoring  $\lambda_n$  and forcing Right to play there, bringing the game to  $\overline{\mathbf{1}}$  with either Left or Right to play next.

If  $n > m \geq 0$  then  $\mathbf{m} + \lambda_n$  is in  $\mathcal{P}^-$  or  $\mathcal{R}^-$ : Left loses as soon as she plays in  $\lambda_n$ , and so plays only in  $\mathbf{m}$ , but (moving first) she will run out of moves in  $\mathbf{m}$  before  $\lambda_n$  is brought to  $\overline{\mathbf{1}}$ . Thus  $\mathbf{n} \not\leq \mathbf{m}$  in this case, since Left can win  $\mathbf{n} + \lambda_n$  but not  $\mathbf{m} + \lambda_n$ .

If  $m < 0$  then let  $k = -m - 1$  and take  $X = \mathbf{k} + \lambda_{n+k}$ . As above,  $\mathbf{n} + \mathbf{k} + \lambda_{n+k} \in \mathcal{L}^-$ . However,  $\mathbf{m} + \mathbf{k} + \lambda_{n+k} \equiv \overline{\mathbf{1}} + \lambda_{n+k} \in \mathcal{N}^-$  since each player can move to a position from which the opponent is forced to move to zero. In this situation we see Left prefers  $\mathbf{n}$  over  $\mathbf{m}$ , so again  $\mathbf{n} \not\leq \mathbf{m}$ .  $\square$

Theorem 5.3.5 tells us that, modulo  $\mathcal{E}$ , the games  $\{\mathbf{0}, \mathbf{1} \mid \cdot\}$  and  $\{\mathbf{0} \mid \cdot\}$  are distinguishable, as the option to  $\mathbf{0}$  does not in general dominate the option to  $\mathbf{1}$ . Thus, in the dead-ending universe, there exist ends that are not integers. However, if we restrict ourselves to sums of *only* dead ends — that is, if there are no non-end dead-ending games around — then the ordering given in Theorem 5.3.4 implies that every end reduces to an integer. Modulo  $\mathcal{E}_e$ , a left option to  $\mathbf{0}$  *does* dominate an option to  $\mathbf{1}$ , and so the games  $\{\mathbf{0}, \mathbf{1} \mid \cdot\}$  and  $\{\mathbf{0} \mid \cdot\}$  are then indistinguishable. The reduction of all ends to integers, modulo dead ends, is given in the following lemma.

**Lemma 5.3.6.** *If  $G$  is a dead end then  $G \equiv \mathbf{n} \pmod{\mathcal{E}_e}$ , where  $n = l(G)$  if  $G$  is a right end and  $n = -r(G)$  if  $G$  is a left end.*

*Proof.* Let  $G$  be a dead right end (the argument for left ends is symmetric). Assume by induction that every option  $G^{L_i}$  of  $G$  (necessarily a dead right end) is equivalent to the integer  $l(G^{L_i})$ . Modulo dead ends, by Theorem 5.3.4, these left options are totally ordered; thus  $G = \{G^{L_1} \mid \cdot\}$  for  $G^{L_1}$  with smallest left-length. Then  $G$  is the canonical form of the integer  $l(G^{L_1}) + 1 = l(G)$ .  $\square$

Lemma 5.3.6 shows that the closure of dead ends has precisely the same monoid as the set of canonical-form integers. The game of DOMINEERING on  $1 \times n$  and  $n \times 1$  strips is an instance of these universes. The results of this section allow us to completely describe the monoid, which is present as Theorem 5.3.7.

**Theorem 5.3.7.** *The misère monoid of the set of normal-play canonical-form integers is*

$$\mathcal{M}_{\mathbb{Z}} = \langle \mathbf{0}, \mathbf{1}, \bar{\mathbf{1}} \mid \mathbf{1} + \bar{\mathbf{1}} = \mathbf{0} \rangle \cong (\mathbb{Z}, +),$$

with outcome partition

$$\mathcal{N}^- = \{\mathbf{0}\}, \mathcal{P}^- = \emptyset, \mathcal{L}^- = \{k\bar{\mathbf{1}} \mid k \in \mathbb{N}\}, \mathcal{R}^- = \{k\mathbf{1} \mid k \in \mathbb{N}\},$$

and total ordering

$$k\mathbf{1} \not\geq j\mathbf{1} \Leftrightarrow k < j,$$

where  $k\mathbf{1} = |k|\bar{\mathbf{1}}$  if  $k < 0$ .

## 5.4 Numbers

### 5.4.1 The Monoid of $\mathbb{Q}_2$ .

A game  $\mathbf{a}$  is a *non-integer number* if it is identical to the normal-play canonical form of a (non-integer) dyadic rational:

$$\mathbf{a} = \frac{m}{2^j} = \left\{ \frac{m-1}{2^j} \mid \frac{m+1}{2^j} \right\},$$

with  $j > 0$  and  $m$  odd. The set of all integer and non-integer (combinatorial game) numbers is thus the set of *dyadic rationals*, which we denote by  $\mathbb{Q}_2$ . As done for integers in the previous section, we now determine the outcome of a general sum of dyadic rationals and thereby describe the misère monoid of the closure of numbers.

Note that the sum of two non-integer numbers (even if both are positive) is not necessarily another number. For example, in general misère play,  $\mathbf{1} + \mathbf{1}/2 = \{\mathbf{1}/2, \mathbf{1} \mid \mathbf{2}\} \neq \mathbf{3}/2$  implies that  $\mathbf{1}/2 + \mathbf{1}/2 = \{\mathbf{1}/2 \mid \mathbf{1} + \mathbf{1}/2\} \neq \mathbf{1}$ . We will see that, unlike integers, the set of dyadic rationals is not closed under disjunctive sum even when restricted to the dead-ending universe; however, closure does occur when we restrict to numbers alone.

Lemma 5.4.2 below — analogous to Lemma 5.3.1 of the previous section — shows that the outcome of a sum of numbers is determined by the left- and right-lengths of the individual numbers. To prove this, we require Lemma 5.4.1. Note that if  $a > 0$  is a dyadic rational then  $l(\mathbf{a}) = 1 + l(\mathbf{a}^L)$ , and if  $a < 0$  is a dyadic rational then  $r(\mathbf{a}) = 1 + r(\mathbf{a}^R)$ . We also have the following inequalities for left-lengths of right options and right-lengths of left options, when  $a$  is a non-integer dyadic rational.

**Lemma 5.4.1.** *If  $a \in \mathbb{Q}_2 \setminus \mathbb{Z}$  is positive then  $l(\mathbf{a}^R) \leq l(\mathbf{a})$ ; if  $a$  is negative then  $r(\mathbf{a}^L) \leq r(\mathbf{a})$ .*

*Proof.* Assume  $a > 0$  (the argument for  $a < 0$  is symmetric). Since  $\mathbf{a}$  is in canonical form, both  $\mathbf{a}^L$  and  $\mathbf{a}^R$  are positive numbers. If  $\mathbf{a}^L = \mathbf{a}^{RL}$  then  $l(\mathbf{a}^R) = 1 + l(\mathbf{a}^{RL}) = 1 + l(\mathbf{a}^L) = l(\mathbf{a})$ . Otherwise  $\mathbf{a}^R = \mathbf{a}^{LR}$ , by Proposition 2.3.1; then  $\mathbf{a}^L$  is not an integer because  $\mathbf{a}^{LR}$  exists, so by induction we obtain  $l(\mathbf{a}^R) = l(\mathbf{a}^{LR}) \leq l(\mathbf{a}^L) = l(\mathbf{a}) - 1 < l(\mathbf{a})$ .  $\square$

We can now determine the outcome of a general sum of numbers, both integer and non-integer.

**Lemma 5.4.2.** *If  $\{a_i\}_{1 \leq i \leq n}$  and  $\{b_i\}_{1 \leq i \leq m}$  are sets of positive and negative numbers, respectively, with  $k = \sum_{i=1}^n l(\mathbf{a}_i) - \sum_{i=1}^m r(\mathbf{b}_i)$ , then*

$$o^- \left( \sum_{i=1}^n \mathbf{a}_i + \sum_{i=1}^m \mathbf{b}_i \right) = \begin{cases} \mathcal{L}^- & \text{if } k < 0 \\ \mathcal{N}^- & \text{if } k = 0 \\ \mathcal{R}^- & \text{if } k > 0. \end{cases}$$

*Proof.* Let  $G = \sum_{i=1}^n \mathbf{a}_i + \sum_{i=1}^m \mathbf{b}_i$ . All followers of  $G$  are also of this form, so assume the result holds for every proper follower of  $G$ . Suppose  $k < 0$ . If  $n = 0$  then Left will run out of moves first because Left cannot move last in any negative number. So assume  $n > 0$ . Left moving first can move in an  $\mathbf{a}_i$  to reduce  $k$  by one (since  $l(\mathbf{a}_i^L) = l(\mathbf{a}_i) - 1$ , which is a left-win position by induction. If Right moves first in an  $\mathbf{a}_i$  then  $k$  does not increase, since  $l(\mathbf{a}_i^R) \leq l(\mathbf{a}_i)$  by Lemma 5.4.1, so the position is a left-win by induction; if Right moves first in a  $\mathbf{b}_i$  then  $k$  does increase by one, but Left can respond in an  $\mathbf{a}_i$  (since  $n > 0$ ) to bring  $k$  down again, leaving another left-win position, by induction. Thus  $G \in \mathcal{L}^-$  if  $k < 0$ .

The argument for  $k > 0$  is symmetric. If  $k = 0$  then either  $G = \mathbf{0}$  is trivially next-win, or both  $n$  and  $m$  are at least 1 and both players have a good first move to change  $k$  in their favour.  $\square$

Lemma 5.4.2 shows that in general misère play, the outcome of a sum of numbers is completely determined by the left-lengths and right-lengths of the positive and negative components, respectively. From this we can conclude that, modulo the closure of canonical-form numbers, a positive number  $\mathbf{a}$  is equivalent to every other number with left-length  $l(\mathbf{a})$ . In particular, every positive number  $\mathbf{a}$  is equivalent to the integer  $l(\mathbf{a})$ . This is Corollary 5.4.3 below; together with Theorem 5.4.5, it will allow us to describe the monoid of canonical-form numbers.

**Corollary 5.4.3.** *If  $\mathbf{a}$  is a number, then*

$$\mathbf{a} \equiv \begin{cases} l(\mathbf{a}) & \text{if } a \geq 0, \\ -r(\mathbf{a}) & \text{if } a < 0. \end{cases}$$

As an example, the dyadic rational  $\mathbf{1/2}$  is equivalent to  $l(\mathbf{1/2}) = \mathbf{1}$ , and  $-\mathbf{3/4} \equiv -r(-\mathbf{3/4}) = -\mathbf{2}$ , modulo  $\mathbb{Q}_2$ . Note that these equivalencies do not hold in the larger universe of  $\mathcal{E}$ ; indeed, as we see in section 4.2, if  $a \neq b$  are numbers then  $a \not\equiv b \pmod{\mathcal{E}}$ .

We see that the closure of numbers is isomorphic to the closure of integers; when restricted to numbers alone, every non-integer is equivalent to an integer. Thus, the misère monoid of numbers is the same monoid presented in Theorem 5.3.7:

$$\mathcal{M}_{\mathbb{Q}_2} = \mathcal{M}_{\mathbb{Z}} = \langle \mathbf{0}, \mathbf{1}, \bar{\mathbf{1}} \mid \mathbf{1} + \bar{\mathbf{1}} = \mathbf{0} \rangle,$$

with the same outcome partition and total ordering.

As with integers, some of the structure found in the number universe is also present in the larger universe  $\mathcal{E}$ . The partial order of the set of numbers, modulo  $\mathcal{E}$ , is described in Section 5.4.2. This subsection ends with a proof that all numbers — not just integers — are invertible in the universe of dead-ending games. The following lemma is required, an extension of Lemma 5.4.2.

**Lemma 5.4.4.** *If  $\{a_i\}_{1 \leq i \leq n}$  and  $\{b_i\}_{1 \leq i \leq m}$  are sets of positive and negative numbers, respectively, and  $\sum_{i=1}^n l(\mathbf{a}_i) - \sum_{i=1}^m r(\mathbf{b}_i) < 0$ , then*

$$o^- \left( \sum_{i=1}^n \mathbf{a}_i + \sum_{i=1}^m \mathbf{b}_i + X \right) = \mathcal{L}^-,$$

for any left end  $X \in \mathcal{E}$ .

*Proof.* The argument from Theorem 5.4.2 works again, since if Right uses his turn to play in  $X$  then Left responds with a move in  $\mathbf{a}_1$  to decrease  $k$  by 1, which is a win for Left by induction.  $\square$

**Theorem 5.4.5.** *If  $a \in \mathbb{Q}_2$  then  $\mathbf{a} + \bar{\mathbf{a}} \equiv \mathbf{0} \pmod{\mathcal{E}}$ .*

*Proof.* Without loss of generality we can assume  $a$  is positive. Since every follower of a number is also a number, we can use Theorem 2.4.2. That is, it suffices to show  $\mathbf{a} + \bar{\mathbf{a}} + X \in \mathcal{L}^- \cup \mathcal{N}^-$  for any left end  $X \in \mathcal{E}$ . If  $X = \mathbf{0}$  this is true by Lemma 5.4.2. If  $X \neq \mathbf{0}$  then we claim  $\mathbf{a} + \bar{\mathbf{a}} + X \in \mathcal{L}^-$ ; assume this holds for all followers of  $\mathbf{a}$ . Left can win playing first on  $\mathbf{a} + \bar{\mathbf{a}} + X$  by moving to  $\mathbf{a}^L$ , since  $l(\mathbf{a}^L) - r(\bar{\mathbf{a}}) = l(\mathbf{a}^L) - l(\mathbf{a}) < 0$  implies  $\mathbf{a}^L + \bar{\mathbf{a}} + X \in \mathcal{L}^-$  by Lemma 5.4.4. If Right plays first in  $X$  then again Left wins by moving  $\mathbf{a}$  to  $\mathbf{a}^L$ ; if Right plays first in  $\bar{\mathbf{a}}$  then Left copies in  $\mathbf{a}$  and wins on  $\mathbf{a}^L + \bar{\mathbf{a}}^L + X \in \mathcal{L}^-$  by induction.  $\square$

Theorem 5.4.5 shows that in dead-ending games like DOMINEERING, HACKENBUSH, etc., any position corresponding to a normal-play canonical-form number has an additive inverse under misère play. So, for example, the positions in Figure 5.2 would cancel each other in a game of misère HACKENBUSH.

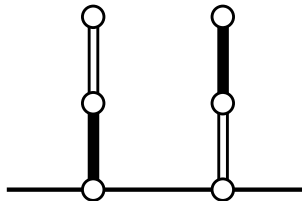


Figure 5.2: Normal-play canonical forms of  $1/2$  and  $-1/2$  in HACKENBUSH.

### 5.4.2 The Partial Order of Numbers Modulo $\mathcal{E}$ .

In Section 5.3, we found that all integers were incomparable in the dead-ending universe. We will see now that non-integer numbers are a bit more cooperative; although not totally ordered, we do have a nice characterization of the partial order of numbers in the universe  $\mathcal{E}$ . First note that any two distinct numbers are distinguishable modulo  $\mathcal{E}$ ; this is an immediate corollary of the following theorem of [6], which extends the result of [19] mentioned before Theorem 5.3.5.

**Theorem 5.4.6.** [6] *If  $G \geq H \pmod{\mathcal{E}}$  then  $G \geq^+ H$ .*

**Corollary 5.4.7.** *If  $a, b \in \mathbb{Q}_2$  and  $a \neq b$  then  $\mathbf{a} \not\equiv \mathbf{b} \pmod{\mathcal{E}}$ .*

Theorem 5.4.6 says that if  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$  then  $a \geq b$  as real numbers (or as normal-play games). The converse is clearly not true for integers, by Theorem 5.3.5; it is also not true for non-integers, since  $\mathbf{1/2} + \overline{\mathbf{1/2}} \in \mathcal{N}^-$  while  $\mathbf{3/4} + \overline{\mathbf{1/2}} \in \mathcal{R}^-$ , so that  $\mathbf{1/2} \not\geq \mathbf{3/4} \pmod{\mathcal{E}}$ . Theorem 5.4.10 shows that the additional stipulation  $l(\mathbf{a}) \leq l(\mathbf{b})$  is sufficient for  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$ . To prove this result we need the following lemmas. As before, non-bolded symbols represent actual numbers, so that ‘ $a < b$ ’ indicates inequality of  $a$  and  $b$  as actual numbers, and  $a^L$  represents the number corresponding to the left-option of the game  $\mathbf{a}$  in canonical form. Recall that if  $\mathbf{x} = \{\mathbf{x}^L | \mathbf{x}^R\}$  is in (normal-play) canonical form then  $\mathbf{x}$  is the *simplest* number (that is, the number with smallest birthday) such that  $x^L < x < x^R$ . Thus, if  $x^L < x, y < x^R$  and  $x \neq y$  then  $\mathbf{x}$  is simpler than  $\mathbf{y}$ .

The first lemma gives a condition on  $a$  and  $b$  to guarantee the left-length of a left option of  $a$  is shorter than the left-length of  $b$ .

**Lemma 5.4.8.** *If  $a$  and  $b$  are positive numbers such that  $a^L < b < a$ , then  $l(\mathbf{a}^L) < l(\mathbf{b})$ .*

*Proof.* We have  $a^L < b < a < a^R$ , so  $\mathbf{a}$  must be simpler than  $\mathbf{b}$ . Thus  $b^L \geq a^L$ , since otherwise  $b^L < a^L < b < b^R$  would imply that  $\mathbf{b}$  is simpler than  $\mathbf{a}^L$ , which is simpler than  $\mathbf{a}$ . Now, if  $b^L = a^L$  then  $l(\mathbf{a}^L) = l(\mathbf{b}^L) = l(\mathbf{b}) - 1 < l(\mathbf{b})$ , and if  $b^L > a^L$  then by induction  $a^L < b^L < b < a$  gives  $l(\mathbf{a}^L) < l(\mathbf{b}^L) = l(\mathbf{b}) - 1 < l(\mathbf{b})$ .  $\square$

Lemma 5.4.8 is used to prove Lemma 5.4.9 below, which is needed for the proof of Theorem 5.4.10 and which strengthens the previous lemma by showing that  $a^L < b <$

$a$  is actually sufficient for  $\mathbf{a} \not\geq \mathbf{b}$ . Note that the following two arguments frequently use the fact that, if  $a \geq b \pmod{\mathcal{E}}$ , then Left wins on the position  $a + \bar{b} + X$  whenever she wins  $X \in \mathcal{E}$ .

**Lemma 5.4.9.** *If  $a$  and  $b$  are positive numbers such that  $a^L < b < a$ , then  $\mathbf{a} \not\geq \mathbf{b} \pmod{\mathcal{E}}$ .*

*Proof.* Note that  $b \notin \mathbb{Z}$  since there are no integers between  $a^L$  and  $a$  if  $\mathbf{a}$  is in canonical form. We must show that Left wins  $\mathbf{a} + \bar{\mathbf{b}} + X$  whenever she wins  $X \in \mathcal{E}$ .

*Case 1:  $b^R = a$ .*

Left can win  $\mathbf{a} + \bar{\mathbf{b}} + X$  by playing her winning strategy on  $X$ . If Right moves in  $\mathbf{a} + \bar{\mathbf{b}}$  to  $\mathbf{a}^R + \bar{\mathbf{b}} + X'$ , then Left responds to  $\mathbf{a}^R + \bar{\mathbf{b}}^R + X' = \mathbf{a}^R + \bar{\mathbf{a}} + X'$ , which she wins by induction since  $a^{RL} \leq a^L$  (see Proposition 2.3.1) gives  $a^{RL} < a < a^R$ . If Right moves to  $\mathbf{a} + \bar{\mathbf{b}}^R + X' = \mathbf{b}^R + \bar{\mathbf{b}}^R + X'$ , with  $X' \in \mathcal{L}^- \cup \mathcal{P}^-$  (since Left is playing her winning strategy in  $X$ ), then Left's response depends on whether  $\mathbf{b}^{RL} = \mathbf{b}^L$  or  $\mathbf{b}^{LR} = \mathbf{b}^R$ : if the former, Left moves to  $\mathbf{b}^{RL} + \bar{\mathbf{b}}^R + X' = \mathbf{b}^L + \bar{\mathbf{b}}^L + X' \equiv X' \pmod{\mathcal{E}}$ ; if the latter then Left moves to  $\mathbf{b}^R + \bar{\mathbf{b}}^{LR} + X' = \mathbf{b}^R + \bar{\mathbf{b}}^R + X' \equiv X'$ . In either case Left wins as the previous player on  $X' \in \mathcal{L}^- \cup \mathcal{P}^-$ .

When Left runs out of moves in  $X$ , she moves to  $\mathbf{a}^L + \bar{\mathbf{b}} + X$ . By Lemma 5.4.8 we know  $l(\mathbf{a}^L) < l(\mathbf{b})$ , and this gives  $\mathbf{a}^L + \bar{\mathbf{b}} + X \in \mathcal{L}^-$  by Lemma 5.4.4.

*Case 2:  $b^R \neq a$ .*

Note that  $b^R$  cannot be greater than  $a$ , since  $a^L < b < a < a^R$  implies  $\mathbf{a}$  is simpler than  $\mathbf{b}$ , while  $b^L < b < a < b^R$  would imply that  $\mathbf{b}$  is simpler than  $\mathbf{a}$ . So  $b^R < a$ , and together with  $a^L < b < b^R$  this gives  $a^L < b^R < a$ , which shows  $\mathbf{a} \geq \mathbf{b}^R \pmod{\mathcal{E}}$  by induction. Similarly  $b^{RL} \leq b^L < b < b^R$  implies  $\mathbf{b}^R \geq \mathbf{b} \pmod{\mathcal{E}}$ , by Case 1. Then by transitivity we have  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$ . □

With Lemma 5.4.9 we can now prove Theorem 5.4.10 below. The corresponding result for negative numbers also holds.

**Theorem 5.4.10.** *If  $a$  and  $b$  are positive numbers such that  $a > b$  and  $l(\mathbf{a}) \leq l(\mathbf{b})$ , then  $\mathbf{a} \not\geq \mathbf{b} \pmod{\mathcal{E}}$ .*

*Proof.* By Corollary 5.4.7 we have  $\mathbf{a} \not\equiv \mathbf{b} \pmod{\mathcal{E}}$ , and so it suffices to show  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$ . Again we have  $b \notin \mathbb{Z}$ . Since  $a > b$ , if  $b > a^L$  then Lemma 5.4.9 gives  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$  as required. So assume  $b \leq a^L$ . Again let  $X \in \mathcal{E}$  be a game which Left wins playing first; we must show Left wins  $\mathbf{a} + \bar{\mathbf{b}} + X$  playing first. Left should follow her winning strategy from  $X$ . If Right plays to  $\mathbf{a} + \bar{\mathbf{b}}^L + X'$ , where  $X' \in \mathcal{L}^- \cup \mathcal{P}^-$ , then Left responds with  $\mathbf{a}^L + \bar{\mathbf{b}}^L + X'$ , which she wins by induction:  $b^L < b \leq a^L$  and  $l(\mathbf{b}^L) = l(\mathbf{b}) - 1 \geq l(\mathbf{a}) - 1 = l(\mathbf{a}^L)$  implies  $\mathbf{a}^L \geq \mathbf{b}^L \pmod{\mathcal{E}}$ .

If Right plays to  $\mathbf{a}^R + \bar{\mathbf{b}} + X'$  (assuming this move exists — that is, assuming  $a \notin \mathbb{Z}$ ) then Left's response is  $\mathbf{a}^{RL} + \bar{\mathbf{b}} + X'$ , if  $a^{RL} > b$ , or  $\mathbf{a}^R + \bar{\mathbf{b}}^R + X'$  if  $a^{RL} \leq b$ . In the first case Left wins by induction because  $a^{RL} > b$  and  $l(\mathbf{a}^{RL}) = l(\mathbf{a}^R) - 1 \leq l(\mathbf{a}) - 1 < l(\mathbf{b})$  implies  $\mathbf{a}^{RL} \geq \mathbf{b} \pmod{\mathcal{E}}$ . In the latter case, note firstly that in fact  $a^{RL} \neq b$ , since we have already seen that as games they have different left-lengths. Then we see  $a^{RL} < b < a < a^R < a^{RR}$ , which shows  $\mathbf{a}^R$  must be simpler than  $\mathbf{b}$ . This gives  $b^R \leq a^R$ , as otherwise  $b^L < b < a < a^r < b^R$  would imply that  $b$  is simpler than  $a^R$ . We can now apply Lemma 5.4.9 to conclude that  $\mathbf{a}^R \geq \mathbf{b}^R \pmod{\mathcal{E}}$ , and so Left wins  $\mathbf{a}^R + \bar{\mathbf{b}}^R + X'$ , with  $X' \in \mathcal{L}^- \cup \mathcal{P}^-$ , as the second player.

Finally, if Left runs out of moves in  $X$  then she moves to  $\mathbf{a}^L + \bar{\mathbf{b}} + X''$  where  $X''$  is a dead left end; then Left wins by Lemma 5.4.4 because  $l(\mathbf{a}^L) < l(\mathbf{a}) \leq l(\mathbf{b}) = r(\bar{\mathbf{b}})$ .  $\square$

**Corollary 5.4.11.** *For positive numbers  $a, b \in \mathbb{Q}_2$ ,  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$  if and only if  $a > b$  and  $l(\mathbf{a}) \leq l(\mathbf{b})$ .*

*Proof.* We must prove the converse of Theorem 5.4.10. Suppose  $a > b$  and  $l(\mathbf{a}) > l(\mathbf{b})$ ; then by Theorem 5.4.6 it cannot be that  $\mathbf{a} \leq \mathbf{b} \pmod{\mathcal{E}}$ , so we need only show  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$ . We have  $\mathbf{b} + \bar{\mathbf{b}} \in \mathcal{N}^-$ , while  $\mathbf{a} + \bar{\mathbf{b}} \in \mathcal{R}^-$ , since in isolation the latter sum is equivalent to the positive integer  $l(\mathbf{a}) - l(\mathbf{b})$ , by Corollary 5.4.3. Thus  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$ .  $\square$

To completely describe the partial order of numbers within  $\mathcal{E}$ , it remains to consider the comparability of  $\mathbf{a}$  and  $\mathbf{b}$  when  $a \geq 0$  and  $b < 0$  (or, symmetrically, when  $a > 0$  and  $b \leq 0$ ). As before, we cannot have  $\mathbf{a} \leq \mathbf{b} \pmod{\mathcal{E}}$ , and the same argument as above ( $\mathbf{b} + \bar{\mathbf{b}} \in \mathcal{N}^-$  and  $\mathbf{a} + \bar{\mathbf{b}} \in \mathcal{R}^-$ ) shows  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$ . The results of this section culminate in Theorem 5.4.12 below.

**Theorem 5.4.12.** *The partial order of  $\mathbb{Q}_2$ , modulo  $\mathcal{E}$ , is given by*

$$\begin{aligned} \mathbf{a} &\equiv \mathbf{b} \pmod{\mathcal{E}} && \text{if } a = b; \\ \mathbf{a} &\geq \mathbf{b} \pmod{\mathcal{E}} && \text{if } 0 < a < b \text{ and } l(\mathbf{a}) \leq l(\mathbf{b}), \\ &&& \text{or } b < a < 0 \text{ and } r(\mathbf{b}) \leq r(\mathbf{a}); \\ \mathbf{a} &\parallel \mathbf{b} \pmod{\mathcal{E}} && \text{otherwise.} \end{aligned}$$

## 5.5 Zeros in the Dead-ending Universe

We have found that integer and non-integer numbers, as well as all ends, satisfy  $G + \overline{G} \equiv \mathbf{0} \pmod{\mathcal{E}}$ . It is not the case that every game in  $\mathcal{E}$  has an additive inverse. For example,  $* + * \not\equiv \mathbf{0} \pmod{\mathcal{E}}$ , because the position  $\mathbf{1}$  occurs in  $\mathcal{E}$ . The equivalence does hold in the dicot universe  $\mathcal{D} \subset \mathcal{E}$ , but this turns out to be of little relevance in the larger universe: none of the day-2 dicots (all but one of which were shown invertible modulo  $\mathcal{D}$  in Chapter 4) are invertible here. This is demonstrated in Proposition 5.5.1 below.

**Proposition 5.5.1.** *No day-2 dicot is invertible in  $\mathcal{E}$ : if  $G \in \{\uparrow, \uparrow*, E, *_2\}$  then  $G + \overline{G} \not\equiv \mathbf{0} \pmod{\mathcal{E}}$ .*

*Proof.*  $*_2$  is immediately non-invertible because  $*_2 + *_2 \not\equiv \mathbf{0}$  even in the smaller universe  $\mathcal{D}$ . To see the other three, we find a game that distinguishes  $G + \overline{G}$  from  $\mathbf{0}$  in each case.

- The integer game  $\mathbf{2}$  is right-win, but in  $\uparrow + \downarrow + \mathbf{2}$ , Left has a good move playing second: Left plays only in the integer, forcing Right to play to zero on his third turn, no matter the sequence of moves.
- The dead-ending position  $X = \{\mathbf{1} \mid \mathbf{0}\}$  is next-win, but Left has a good move playing second in  $\uparrow* + \downarrow* + \{\mathbf{1} \mid \mathbf{0}\}$ . If Right plays in  $\uparrow*$  to zero, Left responds by playing  $\downarrow*$  to zero, and then Right is forced to play  $X$  to zero and lose. If Right plays  $\downarrow*$  to zero, the same strategy works. If Right plays  $\downarrow*$  to  $*$ , then Left plays in  $X$  twice in a row, forcing Right to take the last move in  $\uparrow* + *$ .
- The integer  $\mathbf{1}$  is right-win, but Left has a good move playing first in  $E + \overline{E} + \mathbf{1}$ : Left moves  $E$  to zero, Right has no choice but to move  $\overline{E}$  to  $*$ , and then Left wins playing in  $\mathbf{1}$ .

□

Certainly it is likely that the day-3 dicots invertible in  $\mathcal{D}$  are also non-invertible here. The following lemma describes an infinite family of games that are not invertible in the universe of dead-ending games.

**Lemma 5.5.2.** *If  $G = \{\mathbf{n}_1, \dots, \mathbf{n}_k \mid \overline{\mathbf{n}}_1, \dots, \overline{\mathbf{n}}_k\}$  with each  $n_i \in \mathbb{N}$ , then  $G + \overline{G} \not\equiv \mathbf{0} \pmod{\mathcal{E}}$ .*

*Proof.* Let  $X = \{\mathbf{n}_1, \dots, \mathbf{n}_k \mid \cdot\} \in \mathcal{R}^-$ . Note that  $G = \overline{G}$ . We describe a winning strategy for Left playing second in the game  $G + \overline{G} + X = G + G + X$ . Right has no first move in  $X$ , so Right's move is of the form  $G + \overline{\mathbf{n}}_i + X$ . Left can respond by moving  $X$  to  $\mathbf{n}_i$ , leaving  $G + \mathbf{0}$ . Now Right must play in  $G$  to a nonpositive integer, which as a right end must be in  $\mathcal{L}^-$  or  $\mathcal{N}^-$ . □

The conclusion of these general results for the dead-ending universe is an infinite family of games that are equivalent to zero modulo  $\mathcal{E}$ , which are not all of the form  $G + \overline{G}$  for some  $G$ . These games are illustrated in Figure 5.3. Compare this construction with that of Theorem 4.2.5, which gives a similar family of games that are equivalent to zero modulo  $\mathcal{D}$ .

**Theorem 5.5.3.** *If  $G$  is a dead-ending game such that every  $G^L$  is a left end with an option to zero and every  $G^R$  is a right end with an option to zero, then  $G \equiv \mathbf{0} \pmod{\mathcal{E}}$ .*

*Proof.* Let  $X$  be any game in  $\mathcal{E}$  and suppose Left wins  $X$ . Then Left wins  $G + X$  by following her strategy in  $X$ . If Right plays in  $G$  then she moves to  $G^R + X'$  from a position  $G + X'$  with  $X' \in \mathcal{L}^- \cup \mathcal{P}^-$ ; Left can respond to  $\mathbf{0} + X'$  and win as the second player. If both players ignore  $G$  then eventually Left runs out of moves in  $X$  and plays to  $G^L + X''$ , where  $X''$  is a left end. But  $G^L$  is also a left end, so the sum is a left-win by Lemma 5.2.3. □

## 5.6 PARTIZAN KAYLES

The impartial game KAYLES is played on a row of tokens, with players taking turns removing either a single token or two adjacent tokens. This game has been analyzed

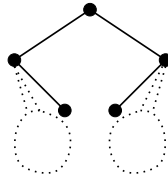


Figure 5.3: An infinite family of games equivalent to zero modulo  $\mathcal{E}$ .

for both normal and misère play [18, 14, 8]. Although there are several natural partizan variations, in this thesis the rule set of PARTIZAN KAYLES is as follows: Left can remove a single token and Right can remove a pair of adjacent tokens. This game can be seen as a one-dimensional variant of DOMINEERING, played on  $1 \times n$  strips, with Left placing the bottom half of her vertical dominoes and Right placing his horizontal dominoes as usual. Note that Left can always move, if the position is non-zero, while Right can move as long as the position is not a sum of single squares. This means there are no non-zero left ends, and any Right end (a sum of single squares) will remain a Right end. Thus, PARTIZAN KAYLES is dead-ending.

This section develops a complete solution for misère PARTIZAN KAYLES, including the misère monoid and partial order of all possible sums of positions. A strip of length  $n$  is denoted  $S_n$ . Let  $\mathcal{K}$  be the set of all possible sums of positions of the form  $S_n$ ,  $n \in \mathbb{Z}^+$ . Note that  $S_0 = \{\cdot \mid \cdot\} = \mathbf{0}$  and  $S_1 = \{\mathbf{0} \mid \cdot\} = \mathbf{1}$  (the normal-play canonical-form integers), but this is not the case for higher values of  $n$ ; in particular  $S_2 = \{\mathbf{1} \mid \mathbf{0}\} \neq \mathbf{2}$ , and, moreover, they are not equivalent in this universe — indeed, Corollary 5.6.4 shows that  $S_2 \not\cong \mathbf{2} \pmod{\mathcal{K}}$ . Note that  $k$  copies of a single square  $S_1$  is equivalent to the canonical integer  $\mathbf{k}$ , since  $kS_1 = \{(k-1)S_1 \mid \cdot\} = \mathbf{k}$ . This shows  $S_2 \not\cong \mathbf{2} = 2S_1$ , since  $S_2 \in \mathcal{P}^-$  while  $2S_1 \in \mathcal{R}^-$ .

It should be immediately apparent to any player of misère games that our partizan version of KAYLES is heavily biased in favour of Right; Left can always move, if the position is non-zero, while Right cannot move on any sum of single squares. It is not surprising, then, that there are no left-win positions in this universe, as demonstrated in Lemma 5.6.1. What is surprising is that there are so many  $\mathcal{N}^-$  and  $\mathcal{P}^-$  positions! We will see that only a third of individual strips  $S_n$  are winnable by Right playing both first and second. For now we prove only the following, which in fact follows from

subsequent results, but is presented here to establish some intuition for the remainder of the section.

**Lemma 5.6.1.** *If  $G$  is a disjunctive sum of one-dimensional strips, with a total of  $n$  empty squares, then*

$$G \in \begin{cases} \mathcal{R}^- \cup \mathcal{N}^- & \text{if } n \equiv 0 \pmod{3} \\ \mathcal{R}^- & \text{if } n \equiv 1 \pmod{3} \\ \mathcal{R}^- \cup \mathcal{P}^- & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

*Proof.* Each of Right's turns reduces the total number of free squares by 2 and each of Left's moves reduces the number by 1. If the total number  $n$  is a multiple of 3 and Right plays first then Left begins each turn with  $3k + 1$  free squares (for some  $k \in \mathbb{N}$ ); in particular Left never begins a turn with zero free squares, and so can never run out of moves before Right. This shows Right wins playing first, so  $G \in \mathcal{R}^- \cup \mathcal{N}^-$ .

If  $n \equiv 1 \pmod{3}$  then Left playing first begins each turn with  $3k + 1$  free squares and Left playing second begins each turn with  $3k + 2$  free squares; in either case Left cannot run out of moves before Right by the same argument as above. Here Right wins playing first or second so  $G \in \mathcal{R}^-$ .

Finally, if the total number of squares is congruent to 2 modulo 3, then Left playing first necessarily moves the game to one in which the total number of squares is congruent to 1 modulo 3, and as shown above this is a Right-win position. Thus Left loses playing first and the game is in  $\mathcal{R}^-$  or  $\mathcal{P}^-$ .  $\square$

Since single squares are so detrimental for Left, we might naively suspect that Left should get rid of them as quickly as she can. That is, given a position that contains an  $S_1$ , Left should do at least as well by playing in the  $S_1$  as playing anywhere else. This is indeed the case, as established in Corollary 5.6.3; the bulk of the work is in the following lemma. It is in the proof of this lemma that the intuition for dead-ending games (or placement games, more narrowly) comes into play: in particular, we use the fact that any piece that can be placed during a current turn could have been placed in a previous turn.

**Lemma 5.6.2.** *If  $G \in \mathcal{K}$  then  $G \geq G^L + S_1 \pmod{\mathcal{K}}$  for all  $G^L \in G^L$ .*

*Proof.* We must show that  $o^-(G + X) \geq o^-(G^L + S_1 + X)$  for any  $X \in \mathcal{K}$ . Since  $G$  is already an arbitrary game in  $\mathcal{K}$ , it suffices to show  $o^-(G) \geq o^-(G^L + S_1)$ .

Fix a left option  $G^{L_0}$  and suppose  $o^-(G^{L_0} + S_1) = \mathcal{N}^-$ , so that Left has a good first move in  $G^{L_0} + S_1$ . If the good move is to  $G^{L_0}$  then Left has a good first move in  $G$ , as desired; otherwise the good move is to  $G^{L_0L} + S_1$ , but then by induction  $o^-(G^{L_0}) \geq o^-(G^{L_0L} + S_1)$  shows that  $G^{L_0}$  is also a good move, and this move is available from  $G$ . Thus  $G \in \mathcal{N}^-$ .

Now suppose  $o^-(G^{L_0} + S_1) = \mathcal{P}^-$ ; we must show that Left has a winning move playing second in  $G$ . Note that  $G^{L_0}$  is not a sum of  $S_1$  positions, since in that case  $G^{L_0} + S_1$  would be right-win. Recall that there are no positions in  $\mathcal{L}^-$ . Since Right has no good first move in  $G^{L_0} + S_1$ , we have  $G^{L_0R} + S_1 \in \mathcal{N}^-$  for all  $G^{L_0R}$ . So Left has a good move from  $G^{L_0R} + S_1$ , and by induction the move to  $G^{L_0R}$  is good. Thus, we have  $G^{L_0R} \in \mathcal{P}^-$  for all Right options of  $G^{L_0}$ . Let  $G^{R_0}$  be any right option. If Right's piece placement from  $G$  to  $G^{R_0}$  does not interfere with Left's move from  $G$  to  $G^{L_0}$ , then Left can place that piece now, leaving  $G^{R_0L}$  equal to  $G^{L_0R} \in \mathcal{P}^-$ . Left wins then as the second player. If Right's move from  $G$  to  $G^{R_0}$  *does* interfere with Left's move from  $G$  to  $G^{L_0}$ , then we have several cases.

Case 1: Right moves from  $G$  to  $G^{R_0}$  by playing in an  $S_n$ ,  $n \geq 3$ . Then Left can play adjacent to Right's piece, and the result is a position identical to some  $G^{L_0R} \in \mathcal{P}^-$ .

Case 2: Right moves from  $G$  to  $G^{R_0}$  by playing in an  $S_2$ , and there is at least one other  $S_2$  in  $G$ . Then Left plays in the other  $S_2$  to leave a position identical to some  $G^{L_0R} \in \mathcal{P}^-$ .

Case 3: Right moves from  $G$  to  $G^{R_0}$  by playing in an  $S_2$ , and there are no other  $S_2$  positions. Then, since  $G^{R_0}$  and  $G^{L_0}$  'overlap',  $G = H + S_2$ ,  $G^{L_0} = H + S_1$ , and  $G^{R_0} = H$ , where  $H$  has no  $S_2$  positions. As noted above,  $H$  is not a sum of  $S_1$  positions, so  $H$  has at least one  $S_n$  with  $n \geq 3$ . Since any  $G^{L_0R} \in \mathcal{P}^-$ , we have  $H^R + S_1 \in \mathcal{P}^-$  for all right options of  $H$ . In particular, Right playing at the end of an  $S_n$ ,  $n \geq 3$ , in  $H$ , leaves a previous-win position  $H' + S_1$ . But Left can construct precisely the same configuration by playing one away from the end of an  $S_n$  in  $G^{R_0} = H$ , leaving  $H' + S_1 \in \mathcal{P}^-$ . This shows Left has a winning response to any  $G^{R_0}$ , and so  $o^-(G) = \mathcal{P}^-$ , as required.  $\square$

As corollaries of this lemma we obtain both a general strategy for Left in PARTIZAN

KAYLES as well as the inequality  $S_2 \not\cong 2S_2$  as foreshadowed above.

**Corollary 5.6.3.** *For any position  $G \in \mathcal{K}$ , if Left can win  $G + S_1$  then Left can win by moving to  $G$ .*

*Proof.* Any other option of  $G + S_1$  is of the form  $G^L + S_1$ , and by Lemma 5.6.2,  $G^L + S_1$  is dominated by  $G$ .  $\square$

**Corollary 5.6.4.**  $S_2 \not\cong S_1 + S_1 \pmod{\mathcal{K}}$ .

*Proof.*  $S_2 \not\cong 2S_1$  follows directly from Lemma 5.6.2 with  $G = S_2$ , since the only left option  $G^L$  is  $S_1$ . The inequality is strict because  $S_2 \in \mathcal{P}^-$  while  $2S_1 \in \mathcal{R}^-$ .  $\square$

This lemma is the key to the solution of PARTIZAN KAYLES: it allows us, by establishing domination of options, to show that every strip  $S_n$  ‘splits’ into a sum of  $S_1$  positions (‘singleton squares’) and  $S_2$  positions (‘dominoes’). Subsequently we determine the outcome of  $kS_1 + jS_2$ . Theorem 5.6.6 demonstrates the reduction, and the remainder of the section deals with outcomes of these sums. Before attempting the argument of Theorem 5.6.6, let us work through a few reductions by hand to gain some insight into this process. These reductions are illustrated in Figure 5.4.

Trivially,  $S_1$  and  $S_2$  are already sums of single squares and dominoes. In a strip of length 3, Left has options to  $S_2$  (playing at either end) and  $S_1 + S_1$  (playing in the middle). Right has only one option, to  $S_1$ . These are precisely the options of  $S_1 + S_2$ ; both games are equal to  $\{S_2, 2S_1 \mid S_1\}$  (which, in canonical form, is the game  $\{S_2 \mid S_1\}$ , by Corollary 5.6.4). Thus,  $S_3 = S_1 + S_2$ .

In a strip of length 4, Left’s options are to  $S_3$  or  $S_1 + S_2$ ; as just established, these are equivalent. Right’s options are to  $S_2$  or  $S_1 + S_1$ , and the second dominates the first by Corollary 5.6.4. So  $S_4 \equiv \{S_1 + S_2 \mid 2S_1\}$ . Compare this to the position  $2S_1 + S_2 = \{S_1 + S_2, 3S_1 \mid 2S_1\}$ ; they are equivalent because the first left option dominates the second. Thus,  $S_4 \equiv 2S_1 + S_2$ .

Lastly, consider a strip of length 5. Left’s options are  $S_4 \equiv 2S_1 + S_2$ ,  $S_1 + S_3 \equiv 2S_1 + S_2$ , and  $S_2 + S_2$ , which dominates the others. Right’s options are  $S_3$  and  $S_1 + S_2$ , which are equivalent. So  $S_5 \equiv \{2S_2 \mid S_1 + S_2\}$ . This is the same as the position  $S_1 + 2S_2$ , as Left’s move to  $2S_2$  dominates here and Right’s only move is to  $S_1 + S_2$ . That is,  $S_5 \equiv S_1 + 2S_2$ .

$$\begin{array}{c}
\boxed{\phantom{0}} \\
\boxed{\phantom{0}}\boxed{\phantom{0}} \\
\boxed{\phantom{0}}\boxed{\phantom{0}}\boxed{\phantom{0}} \equiv \boxed{\phantom{0}} + \boxed{\phantom{0}}\boxed{\phantom{0}} \\
\boxed{\phantom{0}}\boxed{\phantom{0}}\boxed{\phantom{0}}\boxed{\phantom{0}} \equiv \boxed{\phantom{0}} + \boxed{\phantom{0}} + \boxed{\phantom{0}}\boxed{\phantom{0}} \\
\boxed{\phantom{0}}\boxed{\phantom{0}}\boxed{\phantom{0}}\boxed{\phantom{0}}\boxed{\phantom{0}} \equiv \boxed{\phantom{0}} + \boxed{\phantom{0}}\boxed{\phantom{0}} + \boxed{\phantom{0}}\boxed{\phantom{0}}
\end{array}$$

Figure 5.4: Reduction of  $S_n$  into sums of  $S_1$  and  $S_2$ , for  $n = 1, \dots, 5$ .

If we were to continue on with  $S_6, S_7, S_8$ , we would observe a pattern based on the congruency of  $n$  modulo 3. The reductions for longer strips work similarly, and indeed the general inductive proof follows a similar method, of considering the possible options and removing those dominated via Corollary 5.6.4. We now begin the general argument for reducing any  $S_n$ . Lemma 5.6.5 serves to tidy up the proof of Theorem 5.6.6.

**Lemma 5.6.5.** *If  $k, j \in \mathbb{Z}^+$  then  $kS_1 + jS_2 \equiv \{(k-1)S_2 + jS_2 \mid kS_1 + (j-1)S_2\} \pmod{\mathcal{K}}$ .*

*Proof.* Left's only moves in  $kS_1 + jS_2$  are to bring an  $S_1$  to zero or an  $S_2$  to an  $S_1$ . These moves give the options  $(k-1)S_1 + jS_2$  and  $(k+1)S_1 + (j-1)S_2$ , respectively, and the second is dominated by the first because  $S_2 \geq 2S_1$ . Right has only one move up to symmetry — play in an  $S_2$  — and so  $kS_1 + jS_2 \equiv \{(k-1)S_2 + jS_2 \mid kS_1 + (j-1)S_2\} \pmod{\mathcal{K}}$ , as claimed.  $\square$

**Theorem 5.6.6.** *If  $n \geq 3$  then, modulo  $\mathcal{K}$ ,*

$$S_n \equiv \begin{cases} kS_1 + kS_2, & \text{if } n = 3k, \\ (k+1)S_1 + kS_2, & \text{if } n = 3k+1, \\ kS_1 + (k+1)S_2, & \text{if } n = 3k+2. \end{cases}$$

*Proof.* By the lemma, it suffices to show that  $S_n \equiv \{(k-1)S_1 + kS_2 \mid kS_1 + (k-1)S_2\}$  when  $n = 3k$ , that  $S_n \equiv \{kS_1 + kS_2 \mid \{(k+1)S_1 + (k-1)S_2\}$  when  $n = 3k+1$ , and that  $S_n \equiv \{(k-1)S_1 + (k+1)S_2 \mid kS_1 + kS_2\}$  when  $n = 3k+2$ . The proof is broken into these three cases.

Note that any left option of  $S_n$  is of the form  $S_i + S_{n-1-i}$ , with  $0 \leq i \leq n-1$ . Similarly, any right option of  $S_n$  is of the form  $S_i + S_{n-2-i}$ , with  $0 \leq i \leq n-2$ . We will see that, by induction, left options  $S_i + S_{n-1-i}$  and  $S_{i'} + S_{n-1-i'}$  are equivalent if  $i \equiv i' \pmod{3}$ . The same holds for Right's options. Thus, in each of the following three cases, there are at most three distinct left options and three distinct right options, based on the three possibilities for  $i$  modulo 3. Applying induction to these options splits them into sums of  $S_1$  and  $S_2$  positions, and we will see that in every case Corollary 5.6.4 can be used to show that one option dominates the others. In this way we find that in each case  $S_n$  is equivalent to the desired position (as described in the previous paragraph).

*Case 1:  $n = 3k$ :*

If  $i = 3j$  then  $n-1-i = 3k-3j-1 = 3(k-j-1)+2$ , and  $n-2-i = 3k-3j-2 = 3(k-j-1)+1$ . By induction this gives left and right options

$$\begin{aligned} S_i + S_{n-1-i} &= jS_1 + jS_2 + (k-j-1)S_1 + (k-j)S_2 \\ &= (k-1)S_1 + kS_2 = G^{L1}; \text{ and} \\ S_i + S_{n-2-i} &= jS_1 + jS_2 + (k-j)S_1 + (k-j-1)S_2 \\ &= kS_1 + (k-1)S_2 = G^{R1}. \end{aligned}$$

If  $i = 3j+1$  then  $n-1-i = 3k-3j-2 = 3(k-j-1)+1$  and  $n-2-i = 3k-3j-3 = 3(k-j-1)$ . By induction,

$$\begin{aligned} S_i + S_{n-1-i} &= (j+1)S_1 + jS_2 + (k-j)S_1 + (k-j-1)S_2 \\ &= (k+1)S_1 + (k-1)S_2 = G^{L2}; \\ S_i + S_{n-2-i} &= (j+1)S_1 + jS_2 + (k-j-1)S_1 + (k-j-1)S_2 \\ &= kS_1 + (k-1)S_2 = G^{R2}. \end{aligned}$$

If  $i = 3j+2$  then  $n-1-i = 3k-3j-3 = 3(k-j-1)$  and  $n-2-i = 3k-3j-4 =$

$3(k - j - 2) + 2$ , so by induction we have

$$\begin{aligned} S_i + S_{n-1-i} &= jS_1 + (j+1)S_2 + (k-j-1)S_1 + (k-j-1)S_2 \\ &= (k-1)S_1 + kS_2 = G^{L_3}; \\ S_i + S_{n-1-i} &= jS_1 + (j+1)S_2 + (k-j-2)S_1 + (k-j-1)S_2 \\ &= (k-2)S_1 + kS_2 = G^{R_3}. \end{aligned}$$

Left has only two distinct options: either  $G^{L_1} = G^{L_3} = (k-1)S_1 + kS_2$  (obtained by moving to  $S_i + S_{n-1-i}$  with any  $i \equiv 0, 2 \pmod{3}$ ), or  $G^{L_2} = (k+1)S_1 + (k-1)S_2$  (obtained by moving to  $S_i + S_{n-1-i}$  with any  $i \equiv 1 \pmod{3}$ ). By Corollary 5.6.4,  $G^{L_2}$  is dominated by  $G^{L_1}$ . Similarly, Right's options are  $G^{R_1} = G^{R_2} = kS_1 + (k-1)S_2$  or  $G^{R_3} = (k-2)S_1 + kS_2$ , and the latter is dominated by the former. We conclude that, in the case  $n = 3k$ ,

$$S_n \equiv \{(k-1)S_1 + kS_2 \mid kS_1 + (k-1)S_2\} \equiv kS_1 + kS_2.$$

*Case 2:  $n = 3k + 1$ :*

If  $i = 3j$  then  $n - 1 - i = 3k + 1 - 3j - 1 = 3(k - j)$ , and  $n - 2 - i = 3k + 1 - 3j - 2 = 3(k - j - 1) + 2$ . By induction we have

$$\begin{aligned} S_i + S_{n-1-i} &= jS_1 + jS_2 + (k-j)S_1 + (k-j)S_2 \\ &= kS_1 + kS_2, \\ S_i + S_{n-2-i} &= jS_1 + jS_2 + (k-j-1)S_1 + (k-j)S_2 \\ &= (k-1)S_1 + kS_2. \end{aligned}$$

If  $i = 3j + 1$  then  $n - 1 - i = 3k + 1 - 3j - 2 = 3(k - j - 1) + 2$  and  $n - 2 - i = 3k + 1 - 3j - 3 = 3(k - j - 1) + 1$ . By induction,

$$\begin{aligned} S_i + S_{n-1-i} &= (j+1)S_1 + jS_2 + (k-j-1)S_1 + (k-j)S_2 \\ &= kS_1 + kS_2, \\ S_i + S_{n-2-i} &= (j+1)S_1 + jS_2 + (k-j)S_1 + (k-j-1)S_2 \\ &= (k+1)S_1 + (k-1)S_2. \end{aligned}$$

If  $i = 3j + 2$  then  $n - 1 - i = 3k + 1 - 3j - 3 = 3(k - j - 1) + 1$  and  $n - 2 - i = 3k + 1 - 3j - 4 = 3(k - j - 1)$ , so by induction we have

$$\begin{aligned} S_i + S_{n-1-i} &= jS_1 + (j+1)S_2 + (k-j)S_1 + (k-j-1)S_2 \\ &= kS_1 + kS_2, \end{aligned}$$

$$\begin{aligned} S_i + S_{n-1-i} &= jS_1 + (j+1)S_2 + (k-j-1)S_1 + (k-j-1)S_2 \\ &= (k-1)S_1 + kS_2. \end{aligned}$$

In this case, Left's only move is to  $kS_1 + kS_2$ , while Right's option to  $(k-1)S_1 + kS_2$  is dominated by  $(k+1)S_1 + (k-1)S_2$ . Thus, if  $n = 3k + 1$  then

$$S_n \equiv \{kS_1 + kS_2 \mid (k+1)S_1 + (k-1)S_2\} \equiv (k+1)S_1 + kS_2.$$

*Case 3:  $n = 3k + 2$ :*

If  $i = 3j$  then  $n - 1 - i = 3k + 2 - 3j - 1 = 3(k - j) + 1$ , and  $n - 2 - i = 3k + 2 - 3j - 2 = 3(k - j)$ . By induction we have

$$\begin{aligned} S_i + S_{n-1-i} &= jS_1 + jS_2 + (k-j+1)S_1 + (k-j)S_2 \\ &= (k+1)S_1 + kS_2, \end{aligned}$$

$$\begin{aligned} S_i + S_{n-2-i} &= jS_1 + jS_2 + (k-j)S_1 + (k-j)S_2 \\ &= kS_1 + kS_2. \end{aligned}$$

If  $i = 3j + 1$  then  $n - 1 - i = 3k + 2 - 3j - 2 = 3(k - j)$  and  $n - 2 - i = 3k + 2 - 3j - 3 = 3(k - j - 1) + 2$ . By induction,

$$\begin{aligned} S_i + S_{n-1-i} &= (j+1)S_1 + jS_2 + (k-j)S_1 + (k-j)S_2 \\ &= (k+1)S_1 + kS_2, \end{aligned}$$

$$\begin{aligned} S_i + S_{n-2-i} &= (j+1)S_1 + jS_2 + (k-j-1)S_1 + (k-j)S_2 \\ &= kS_1 + kS_2. \end{aligned}$$

If  $i = 3j + 2$  then  $n - 1 - i = 3k + 2 - 3j - 3 = 3(k - j - 1) + 2$  and  $n - 2 - i = 3k + 1 - 3j - 4 = 3(k - j - 1) + 1$ , so by induction we have

$$\begin{aligned} S_i + S_{n-1-i} &= jS_1 + (j+1)S_2 + (k-j-1)S_1 + (k-j)S_2 \\ &= (k-1)S_1 + (k+1)S_2, \\ S_i + S_{n-1-i} &= jS_1 + (j+1)S_2 + (k-j)S_1 + (k-j-1)S_2 \\ &= kS_1 + kS_2. \end{aligned}$$

In this case, Left's move to  $(k+1)S_1 + kS_2$  is dominated by her move to  $(k-1)S_1 + (k+1)S_2$ , while Right's only option is  $kS_1 + kS_2$ . Thus if  $n = 3k + 2$  then

$$S_n \equiv \{(k-1)S_1 + (k+1)S_2 \mid kS_1 + kS_2\} \equiv kS_1 + (k+1)S_2.$$

□

We have shown that every strip splits into a sum of singleton squares and dominoes. This makes the analysis of the PARTIZAN KAYLES universe much more manageable; we need only determine the outcome of a sum of any number of singleton squares and dominoes. One obvious observation is that if there are more single squares than dominoes, then Left will not be able to win, as Right can eliminate all of 'his' pieces before Left can run out of squares. That is, if  $k > j$  then  $o^-(kS_1 + jS_2) = \mathcal{R}^-$ . Another immediate result is the outcome when there are exactly as many single squares as dominoes; the players are forced<sup>2</sup> into a Tweedledum-Tweedledee situation where the first player runs out of moves first. Thus, if  $k = j$  then  $o^-(kS_1 + jS_2) = o^-(kS_1 + kS_2) = \mathcal{N}^-$ . The outcome in the remaining case, when  $k < j$ , turns out to be dependant on the congruence of the total number of (not necessarily single) squares, modulo 3; that is, it depends on the value of  $k + 2j \pmod{3}$ . This is not surprising in light of the very first result of this section, Lemma 5.6.1. Indeed, the following theorem has that lemma as a consequence.

---

<sup>2</sup>The players are 'forced' under optimal play, because Left will always choose to play in an  $S_1$  over an  $S_2$ , by Corollary 5.6.3.

**Theorem 5.6.7.** For positive integers  $k$  and  $j$ ,

$$o^-(kS_1 + jS_2) = \begin{cases} \mathcal{N}^-, & \text{if } k = j, \text{ or if } k < j \text{ and } k + 2j \equiv 0 \pmod{3}, \\ \mathcal{R}^-, & \text{if } k > j, \text{ or if } k < j \text{ and } k + 2j \equiv 1 \pmod{3}, \\ \mathcal{P}^-, & \text{if } k < j \text{ and } k + 2j \equiv 2 \pmod{3}. \end{cases}$$

*Proof.* Lemma 5.6.5 states that

$$kS_1 + jS_2 \equiv \{(k-1)S_1 + jS_2 \mid kS_1 + (j-1)S_2\}.$$

Let  $G = kS_1 + jS_2$ . We can prove each case by applying induction to  $G^L = (k-1)S_1 + jS_2$  and  $G^R = kS_1 + (j-1)S_2$ .

If  $k = j$  then Left's option is in  $\mathcal{P}^-$  since  $(k-1) + 2k = 3k - 1$ , and Right's option is in  $\mathcal{R}^-$  since  $k > k - 1$ . So  $G \in \mathcal{N}^-$ .

If  $k > j$  then  $G^L \in \mathcal{N}^- \cup \mathcal{R}^-$  and  $G^R \in \mathcal{R}^-$ , so  $G \in \mathcal{R}^-$ .

If  $k < j$  and  $k + 2j \equiv 0 \pmod{3}$  then  $G^L \in \mathcal{P}^-$  because  $k - 1 + 2j \equiv 2 \pmod{3}$ , while  $G^R \in \mathcal{R}^-$  because  $k + 2j - 2 \equiv 1 \pmod{3}$ . Thus  $G \in \mathcal{N}^-$ .

If  $k < j$  and  $k + 2j \equiv 1 \pmod{3}$  then  $G^L \in \mathcal{N}^-$  because  $k - 1 + 2j \equiv 0 \pmod{3}$ , and  $G^R \in \mathcal{P}^-$  because  $k + 2j - 2 \equiv 2 \pmod{3}$ . This confirms  $G \in \mathcal{R}^-$ , which in fact we already knew from Lemma 5.6.1.

Finally, if  $k < j$  and  $k + 2j \equiv 2 \pmod{3}$  then  $G^L \in \mathcal{R}^-$  because  $k - 1 + 2j \equiv 1 \pmod{3}$ , and  $G^R \in \mathcal{N}^-$  because  $k + 2j - 2 \equiv 0 \pmod{3}$ . Thus  $G \in \mathcal{P}^-$ .  $\square$

We now know the outcome of an arbitrary PARTIZAN KAYLES; given a disjunctive sum of strips of various length, we can split each into single squares and dominoes, and then apply Theorem 5.6.7 to determine the outcome. It would be nice to go a few steps further and answer the following questions.

1. Can we 'look' at a general sum of strips and determine the outcome, without having to first reduce the position to single squares and dominoes?
2. Can we determine the optimal move for a player when he or she has a winning strategy?

The following corollaries to Theorem 5.6.7 will help answer these questions. The first shows what might intuitively be guessed in this universe: a single square and a

single domino ‘cancel each other out’. Essentially, we can think of a single square as one move for Left and a single domino as one move for Right. Things are more complicated when only dominoes are present, because Left must then play in a domino, but this way of thinking works when at least one of each exists.

**Corollary 5.6.8.**  $S_1 + S_2 \equiv \mathbf{0} \pmod{\mathcal{K}}$ .

*Proof.* Let  $X \equiv kS_1 + jS_2$  be any Kayles sum. Then  $o^-(X + S_1 + S_2) = o^-[(k + 1)S_1 + (j + 1)S_2] = o^-(kS_1 + jS_2)$ , by Theorem 5.6.7, since

$$k = j \Leftrightarrow k + 1 = j + 1,$$

$$k > j \Leftrightarrow k + 1 > j + 1, \text{ and}$$

$$k + 2j \equiv (k + 1) + 2(j + 1) \pmod{3}.$$

□

At this point we can identify the monoid of PARTIZAN KAYLES positions. After reducing using equivalency and the above corollary, every position in  $\mathcal{K}$  is of the form  $kS_1$ , for an integer  $k$ , where  $kS_1 = |k|S_2$  if  $k < 0$ . The monoid is thus a group, isomorphic to the integers, as many of the monoids of this thesis have been. We have

$$\mathcal{M}_{\mathcal{K}} = \langle \mathbf{0}, S_1, S_2 \mid S_1 + S_2 = \mathbf{0} \rangle,$$

with outcome partition

$$\mathcal{N}^- = \{kS_2 \mid k \geq 0, k \equiv 0 \pmod{3}\}, \mathcal{P}^- = \{kS_2 \mid k > 0, k \equiv 1 \pmod{3}\},$$

$$\mathcal{R}^- = \{jS_1, kS_2 \mid j > 0, k > 0, k \equiv 2 \pmod{3}\}, \mathcal{L}^- = \emptyset.$$

Corollary 5.6.8 has a very nice obvious consequence, which is given as the next corollary: any strip of length a multiple of 3 is equivalent to zero. This is a partial answer to our first question, because in a general sum of strips players can simply ignore any strips of such length.

**Corollary 5.6.9.** *If  $n \equiv 0 \pmod{3}$  then  $S_n \equiv \mathbf{0} \pmod{\mathcal{K}}$ .*

*Proof.* This is clear from Theorem 5.6.6 and the previous corollary, since if  $n = 3k$  then  $S_n$  reduces to  $kS_1 + kS_2 = k(S_1 + S_2) \equiv \mathbf{0} \pmod{\mathcal{K}}$ . □

The next theorem precisely answers question 1, by describing the outcome of a general Kayles position *without* directly computing its reduction into  $S_1$  and  $S_2$  pieces. We must simply compare the number of pieces of length congruent to 1 modulo 3 to the number of those congruent to 2 modulo 3. In fact, there is no new argument here: this is a compression of the two steps already discussed — the reduction into  $S_1$  and  $S_2$  pieces (Theorem 5.6.6) and the outcome of  $kS_1 + jS_2$  (Theorem 5.6.7).

**Theorem 5.6.10.** *If  $G$  is a PARTIZAN KAYLES position with  $x$  pieces of length 1 modulo 3 and  $y$  pieces of length 2 modulo 3, then*

$$o^-(G) = \begin{cases} \mathcal{N}^-, & \text{if } x = y, \\ & \text{or if } x < y \text{ and } x + 2y \equiv 0 \pmod{3}; \\ \mathcal{R}^-, & \text{if } x > y, \\ & \text{or if } x < y \text{ and } x + 2y \equiv 1 \pmod{3}; \\ \mathcal{P}^-, & \text{if } x < y \text{ and } x + 2y \equiv 2 \pmod{3}. \end{cases}$$

Finally, Theorem 5.6.11 answers our second question, of most interest to any actual player of PARTIZAN KAYLES: *how* do you win a general non-reduced PARTIZAN KAYLES position, when you can?

**Theorem 5.6.11.** *If Left can win a PARTIZAN KAYLES position, then she can win playing at the end of a strip of length 1 modulo 3, when possible, or the end of a strip of length 2 modulo 3, otherwise. If Right can win a PARTIZAN KAYLES position, then he can win playing at the end of a strip of length 2 modulo 3, when possible, or one away from the end of a strip of length 1 modulo 3, otherwise.*

*Proof.* Let  $G = (x, y)$  be a PARTIZAN KAYLES position with  $x$  pieces of length 1 modulo 3 and  $y$  pieces of length 2 modulo 3. Note that Left playing at the end of one of the  $x$  strips leaves a strip equivalent to zero, thereby moving  $G = (x, y)$  to  $(x - 1, y)$ . Right playing at the end of a  $y$  strip similarly moves  $G$  to  $(x, y - 1)$ .

If  $x = y \neq 0$  (so  $G \in \mathcal{N}^-$ ), Left's move to  $(x - 1, y)$  is in  $\mathcal{P}^-$ , by Theorem 5.6.10, since  $x - 1 < y$  and  $x - 1 + 2y = x - 1 + 2x \equiv 2 \pmod{3}$ . Right's move to  $(x, y - 1)$  is in  $\mathcal{R}^-$  because  $x > y - 1$ .

If  $x > y \neq 0$  then Right's move to  $(x, y - 1)$  is in  $\mathcal{R}^-$  as above. If  $x > y = 0$  then either Right has no move (if all  $x$  pieces are singleton squares), or Right can play

one away from the end of a strip of length 1 modulo 3 to leave two strips of length 1 modulo 3. This gives a position  $(x + 1, y)$ , which is in  $\mathcal{R}^-$  since  $x + 1 > y$ .

If  $x < y$  and  $x + 2y \equiv 0 \pmod{3}$ , then Left's move to  $(x - 1, y)$  is in  $\mathcal{P}^-$ , if it exists, because  $x - 1 + 2y \equiv 2 \pmod{3}$ . If  $x = 0$  then  $y > 0$  and Left can move at the end of a strip congruent to 2 modulo 3 to leave  $(x + 1, y - 1) = (1, y - 1)$ . It cannot be that  $1 = y - 1$  because then  $x + 2y = 0 + 2 \not\equiv 0 \pmod{3}$ . So  $x + 1 < y - 1$  and  $x + 1 + 2(y - 1) = x + 2y - 1 \equiv 2 \pmod{3}$ , which means this back-up move of Left's is in  $\mathcal{P}^-$ . Right can play at the end of a strip congruent to 2 modulo 3 and move to  $(x, y - 1)$ . It cannot be that  $x = y - 1$  since then  $x + 2y = y - 1 + 2y \not\equiv 0 \pmod{3}$ , so  $x < y - 1$  and  $x + 2(y - 1) \equiv 2 \pmod{3}$ , which means this move is in  $\mathcal{P}^-$ .  $\square$

The universe of PARTIZAN KAYLES is unique among all other monoids of this thesis, for several reasons. Firstly, it is not closed under conjugates. Secondly, it contains an instance of  $G + H \equiv \mathbf{0}$  without  $H \equiv \overline{G}$ :  $S_1$  and  $S_2$  are 'additive inverses' modulo  $\mathcal{K}$ , but neither is the conjugate of the other. This serves as a counterexample to a stronger version of Conjecture 2.1.7, and shows that we do at least need the universe to be closed under conjugates for that conjecture to be true. A related point of interest is that we have in  $\mathcal{K}$  an example of a position in  $\mathcal{R}^-$  that is the inverse of a position in  $\mathcal{P}^-$ . This is very bizarre indeed!

## 5.7 Further Potential for Dead-ending Games

This chapter has developed some initial results for the universe of dead-ending games, including the monoids for several obvious subuniverses — ends, numbers — as well as the specific game of PARTIZAN KAYLES. The dead-ending condition circumvents many of the limitations of general misère analysis, and its significance is even more striking when we consider how naturally it arises as a property of many common games. Such games mostly also belong to the family of placement games, which is emerging as a very interesting subuniverse of dead-ending games.

Recall our definition of a *placement game* as one in which pieces are placed on a board in such a way that a player's available placements at any time are a subset of those from the previous turn. Michael Albert has noted [personal communication, 2012] that most of these games can be thought of as games on graphs: every possible

location for a Left piece is a blue vertex, every possible location for a Right piece is a red vertex, and two vertices are adjacent if they cannot both be played (for example, if the corresponding pieces would overlap). A move is to choose a vertex of your colour and delete all adjacent vertices. With this representation, placement games can be analyzed in terms of the underlying graph structure; for example, it may be possible to fully understand such games on paths or on trees.

Clearly there is great potential for further investigation into dead-ending and placement games. A natural extension of this work would also be to analyze other specific games in the context of the dead-ending or placement universe, as done for *PARTIZAN KAYLES* in the previous section. Games such as *NOGO*, *COL*, and *SNORT* would be excellent candidates.

## Chapter 6

### Conclusion

This thesis has examined three main universes of misère games, each defined by imposing a different game-play restriction. In some sense, the alternating property of Chapter 3 is the very opposite of the dicot property of Chapter 4: every option of an alternating game is an end, while no non-zero option of a dicot game can be an end. The dead-ending property, which generates a superset of the dicot games, is likewise far removed from alternating games. No game born after day 1 can satisfy the definition of both an alternating end and a dead end: the former requires that followers alternate between left ends and right ends, and the latter forbids it. Dead-ending games, though including the dicots, also form a much different universe than the dicot universe; this is at least partially due to the presence of integers and other numbers, the absence of which is key to many of the ‘nice’ properties of dicot games, including the invertibility of day 1 and day 2 games.

Despite these distinctions, there are several interesting commonalities between all three universes. In every chapter, there arises a monoid isomorphic to the integers or to a direct product of the integers: the subuniverse of alternating ends ( $\mathbb{Z}$ ) in Chapter 3, the closures of day-2 dicot inverse pairs ( $\mathbb{Z} \times \mathbb{Z}_2$ ) in Chapter 4, and ends, integers, numbers, and PARTIZAN KAYLES positions ( $\mathbb{Z}$ ) in Chapter 5. Given the above description of alternating ends and dead ends as having directly opposing properties, it is remarkable that their monoids are in fact isomorphic. It suggests that how games relate to each other inside a universe, more so than the common defining property, is what determines the algebraic structure of a universe. Note that not *all* infinite monoids of this thesis are simply the group of integers: the monoid  $\mathcal{M}_A$  of all alternating positions, as a proper superset of  $\mathcal{M}_{A_e}$ , is not isomorphic to  $(\mathbb{Z}, +)$ , nor is the monoid  $\mathcal{M}_S$  of HACKENBUSH SPRIGS.

Equivalency to zero has also been a common thread throughout this thesis. Beginning with Theorem 2.4.2 of Chapter 2, which establishes invertibility criteria for

any universe, conditions for  $G \equiv 0$  or  $G + \overline{G} \equiv 0$  crop up in almost every section. The structure of integers, in particular, in each of these universes, is very promising. Since every integer is a sum of alternating ends, Chapter 3 found that integers are invertible modulo alternating games; Chapter 4 found that star-based integers and numbers are invertible modulo dicot games; and Chapter 5 found that all numbers are invertible modulo dead-ending games.

Finally, a more subtle link between the alternating games, dicots (more specifically HACKENBUSH SPRIGS), and dead-ending games, is the relationship between the concepts of *tilt*, *advantage*, and *left-length*. As discussed in Section 4.5, the way in which the tilt of a sum of alternating ends affects the outcome of a sum, even a sum that includes a non-end alternating game, is similar to the way in which a player's advantage in a sum of Sprigs affects the outcome. If the tilt of  $X$  is tipped far enough in Left's favour, then Left wins  $X + G$ , regardless of the outcome of  $G$ , and it is conjectured that this would also be true of  $X + G_1 + \dots + G_n$  for any number of non-end alternating games  $G_i$ . If the tilt is in a certain interval around zero, which would increase as  $n$  increases, then the outcomes of the individual games  $G_i$  would come into play. Similarly, if the *advantage*  $\Delta$  is larger than 1, then Left wins the Sprig sum  $(X, Y) + *$  regardless of the specific numbers  $x_i, y_i \in X, Y$ ; if the advantage is within 1 of 0, however, then the *lean*, or difference in minimum values of  $X$  and  $Y$ , must be considered.

The functions *left-length* and *right-length*, defined in Chapter 5 for dead-ending games, work for normal-play numbers in the way that tilt works for alternating ends and advantage works for a Sprig sum  $(X, Y)$ . Each assigns to a sum a numerical value that completely determines the outcome, with a value of zero indicating a next-win sum, and positive or negative values indicating left- or right-win sum (see Theorem 3.2.10, Theorem 4.5.9, and Lemma 5.4.2). In these cases, unlike for  $X + G$  or  $(X, Y) + *$  above, no other considerations are necessary. It would be interesting to see if the work of Chapter 5 could be extended to include some superset of the normal-play numbers; the outcome of such a sum might be determined by left-length or right-length if they are 'extreme' enough, and be determined by some other factor if the left-length or right-length are close to zero.

As outlined in the concluding section of each of the previous three chapters, there

are a number of ways to naturally extend this research. The most interesting and most promising open problems are described below.

- A proof for Conjecture 2.1.7, which asserts that only  $\overline{G}$  can be the additive inverse of  $G$  in a conjugate-closed universe, would fill a significant hole in the theory of indistinguishability quotients. It seems that an inductive argument (by contradiction) should follow from considering a minimal counterexample, but perhaps additional techniques will have to be developed in order for this to work.
- It would be nice to extend Theorem 3.4.3 (which gives the outcome of  $G + X$  for  $G$  a non-end alternating game and  $X$  a sum of alternating ends), as described above in the discussion of tilt. For example, it may be possible, though tedious, to find the outcome of  $G + H + X$ , where  $G$  and  $H$  are non-end alternating games and  $X$  is a sum of alternating ends.
- Among the many alternating, dicot, or dead-ending games that could now be analyzed with the help of general results from Chapters 3, 4, and 5 (as done for PENNY NIM, HACKENBUSH SPRIGS, and PARTIZAN KAYLES), the game of DOMINEERING is of particular interest. The so-called ‘one-dimensional’ version is now solved, in the form of PARTIZAN KAYLES, and some progress and conjectures have been made for the  $2 \times n$  game. Given that DOMINEERING is not only a dead-ending game, but also a *placement game*, there is great potential for analyzing  $2 \times n$  or other restricted positions.
- Determining the monoid of all day-2 dicots, or at least all invertible day-2 dicots (that is, all but  $*_2$ ), seems a very reasonable goal. There are only two options of all these games — zero and  $*$  — and the results of Section 4.3 show how easily the outcome of a sum of any inverse pair can be described. A very attainable first step would be to determine the closure of  $\{\uparrow, \downarrow, \uparrow*, \downarrow*\}$ .
- One way to define a *placement game* is to impose the following restriction on the game board: all subsets of a legal board position (that is, all subsets of piece placements) must also be legal. Given that this is a stronger condition than dead-ending, we would expect many nice general results to emerge. There

is also the option of exploring placement games using Michael Albert's graph representation; here a first step would be to consider the game on paths or other basic graphs.

Of all of these ideas, narrowing the focus from dead-ending games to the subuniverse of placement games is probably the most exciting and natural future direction for the present research. For *misère* play, analysis of the universe of placement games will begin between the authors of [12] in April 2013.

## Appendix A

### Rule Sets

- **COL** is a variant of SNORT; both games are played on a graph with each vertex uncoloured, coloured black, or coloured white. In COL, Left moves by choosing an uncoloured vertex not adjacent to any black vertices and colouring it black. Right moves by choosing an uncoloured vertex not adjacent to a white vertex and colouring it white.
- **DOMINEERING** is played on a grid of squares, some of which may be missing (already played). Left places a vertical domino to remove two adjacent vertical squares, and Right places a horizontal domino to remove two adjacent horizontal squares.
- **HACKENBUSH** is played on a graph with edges coloured blue (black), red (white), and green (gray). One vertex is called the *ground* and is indicated by a horizontal line. Left cuts a blue or green edge and removes any edges of the graph that are no longer connected to the ground; Right cuts a red or green edge and likewise removes any portion now unconnected to the ground.
- **HEX** is played on an  $n \times n$  hexagonally-tiled board. Left places a black stone on any empty hexagon, aiming to connect the upper-left side and the lower-right side of the board with a path of black stones. Right places white stones and tries to connect the lower-left and upper-right.
- **KAYLES** is an impartial game played on one or more rows of pins. A player moves by knocking down any single pin or two adjacent pins.
- **KONANE** is played on a rectangular board, usually with alternating black and white stones in a checkerboard pattern, and with two adjacent stones removed from the centre. Left can jump a black stone over an orthogonally adjacent white stone onto an empty square, and Right similarly jumps a white stone

over an adjacent black stone. Multiple jumps in the same direction may be chained together in a single turn.

- **MANCALA** is played on a game board that has some number of ‘pits’ on each side of the board, as well as one pit for each player at the ends of the board. The game begins with a number of stones in each of the non-end pits. Left moves by picking up all stones from one pit and dropping one stone into each of the following (counterclockwise) pits. There are various conditions under which Left may capture stones on the opponent’s side of the board. Right moves similarly, and the game ends when all stones are in the end pits; the winner is the player with the most stones in his or her end.
- **MAZE** is played on a rectangular grid tilted at a 45-degree angle, with some edges highlighted. A token starts on some square (often the top square), and Left can move it down and to the left, while Right can move it down and to the right. Neither player can move the token through a highlighted edge.
- **NIM** is an impartial game played on one or more heaps of tokens. A player chooses a heap and removes at least one token from that heap.
- **NOGO** is played on the intersections of a grid. Left places a black stone on an empty intersection and Right places a white stone on an empty intersection, so that after each placement, every connected group of white stones and every connected group of black stones is adjacent to at least one empty intersection. In the language of the game GO, neither player is allowed to *capture* the opponent or to self-capture (commit suicide).
- **SNORT** is played on a graph with each vertex uncoloured, coloured black, or coloured white. Left moves by choosing an uncoloured vertex that is not adjacent to a white vertex and colouring it black; Right symmetrically chooses an uncoloured vertex not adjacent to black and colours it white.

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