

BOUNDED STABLE SETS: POLYTOPES AND COLORINGS*

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Abstract. A k -stable set in a graph is a stable set of size at most k . We study the convex hull of the k -stable sets of a graph, aiming for a complete inequality description. We also consider colorings of weighted graphs by k -stable sets, aiming for a relation between the values of an optimal coloring and an optimal fractional coloring. Results for $k = 2$ and $k = 3$ as well as a number of general conjectures linking fractional and integral colorings are given.

Key words. fractional graph coloring, bounded coloring, stable set polytope

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1. The results. Let $G = (V, E)$ be a graph. A *stable set* of G is a set of mutually nonadjacent nodes of G . The *stable set polytope* of G is the convex hull of incidence vectors of its stable sets.

Let $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$. An (integral) *coloring* of G with respect to \mathbf{c} , or simply a coloring of (G, \mathbf{c}) , is an assignment ϕ of colors to the nodes of G such that

- $\phi(v)$ is a set of $c(v)$ colors, for all $v \in V(G)$, and
- $\phi(v) \cap \phi(u) = \emptyset$ for any two adjacent nodes v and u of G .

Let \mathcal{J} be the family of stable sets of G . In terms of vectors, a coloring of (G, \mathbf{c}) is any element of

$$P_I(G, \mathbf{c}) = \left\{ \mathbf{y} \in \mathbb{Z}_+^{\mathcal{J}} : \sum_{v \in S \in \mathcal{J}} y_S = c(v), v \in V(G) \right\}.$$

A *fractional coloring* of (G, \mathbf{c}) is any element of

$$P(G, \mathbf{c}) = \left\{ \mathbf{y} \in \mathbb{Q}_+^{\mathcal{J}} : \sum_{v \in S \in \mathcal{J}} y_S = c(v), v \in V(G) \right\}.$$

For any given coloring \mathbf{y} , fractional or integral, those stable sets S for which $y_S > 0$ are its *color classes*. The interest is in finding colorings of (G, \mathbf{c}) that use as few colors as possible. The number of colors used by any optimal (integral) coloring of (G, \mathbf{c}) , known as the *chromatic number* of (G, \mathbf{c}) , is denoted by $\chi(G, \mathbf{c})$. In other words,

$$\chi(G, \mathbf{c}) = \min\{\mathbf{1} \cdot \mathbf{y} : \mathbf{y} \in P_I(G, \mathbf{c})\},$$

where $\mathbf{1}$ is the all-one row vector. The *fractional chromatic number* of (G, \mathbf{c}) , denoted $\eta(G, \mathbf{c})$, is the number

$$\eta(G, \mathbf{c}) = \min\{\mathbf{1} \cdot \mathbf{y} : \mathbf{y} \in P(G, \mathbf{c})\}.$$

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A k -stable set of G is a stable set of cardinality at most k . By letting \mathcal{J} be the family of k -stable sets of G , we have the analogous notion of a coloring (fractional or integral) by k -stable sets. The corresponding values of optimal colorings will be denoted by $\chi_k(G, \mathbf{c})$ and $\eta_k(G, \mathbf{c})$. The convex hull of incidence vectors of k -stable sets of G will be referred to as the k -stable set polytope of G . This is a polytope of full dimension in $\mathbb{Q}^{V(G)}$, since all unit vectors and the zero vector correspond to k -stable sets of G .

There are two related problems we deal with in this paper. The first involves “slicing off” the stable set polytope of a graph G with the inequality $\sum_{v \in V(G)} x(v) \leq k$, for some positive integer k , and studying the remaining polytope. The integral points of this polytope correspond to the k -stable sets of G , and we are interested in the inequalities that define their convex hull.

For $k = 2$ we give a complete inequality description of the k -stable set polytope of any graph. For $k = 3$ we give an analogous result for the k -stable set polytope of bipartite graphs. In either case, the system described is totally dual integral and each inequality is facet defining.

The second problem involves the study, for any $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$, of the relationship between integral and fractional colorings by k -stable sets. A description of the k -stable set polytope of a graph G gives a formula for $\eta_k(G, \mathbf{c})$, the value of an optimal fractional coloring of (G, \mathbf{c}) by k -stable sets for any $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$. This formula can be used as a starting point for studying (integral) colorings of (G, \mathbf{c}) by k -stable sets. In particular, when $\chi_k(G, \mathbf{c}) = \lceil \eta_k(G, \mathbf{c}) \rceil$, the corresponding polytope yields necessary and sufficient conditions on the colorability of G by k -stable sets. In general, it is desirable to obtain upper bounds on $\chi_k(G, \mathbf{c})$ expressed as functions of $\eta_k(G, \mathbf{c})$.

For $k = 2$ we show that $\chi_k(G, \mathbf{c}) \leq \eta_k(G, \mathbf{c}) + \frac{\eta(G)}{2}$, where $\eta(G) = \eta(G, \mathbf{1})$. For $k = 3$ we show that $\chi_k(G, \mathbf{c}) \leq \lceil \eta_k(G, \mathbf{c}) \rceil + 1$ when G is bipartite. In terms of polytopes, and in view of the results of Baum and Trotter, Jr. [2], the first inequality shows that for every vector \mathbf{x} in the 2-stable set polytope of G and every $p \in \mathbb{Z}_+$, $p\mathbf{x}$ can be written as the sum of $\lceil \eta_k(G, \mathbf{c}) + \frac{\eta(H)}{2} \rceil$ incidence vectors of 2-stable sets. An analogous statement holds also for the second result.

Colorings by k -stable sets, or, as they are better known, k -bounded node colorings, arise in chromatic scheduling problems when the number of rooms, machines, or other resources is limited (see [11]). Chen and Lih [4] have established a formula for the k -bounded chromatic number of an unweighted tree. Lower bounds, upper bounds, and complexity results on k -bounded colorings are given by Hansen, Hertz, and Kuplinsky [7].

For colorings in general, there is no relationship between the fractional and integral chromatic number of a graph. In [12] a family of graphs G is given such that $\eta(G) \rightarrow 2$, whereas $\chi(G) \rightarrow \infty$ as $|V(G)| \rightarrow \infty$. However, for special classes of graphs, relationships between fractional and integral colorings have been established even for weighted graphs (see [10]). Colorings of line graphs (or edge colorings) in particular have attracted great attention. Several high profile conjectures attempt to link fractional and integral colorings in these graphs, but so far the results have been limited. We refer the reader to [9] and [8] for further explanations on the problems involved.

The stable set polytope of graphs G with stability number at most 2 has been described by Cook [5]. A proof of this result is also given in [15]. The stable set polytope in general has been studied extensively. The reader is referred to [16] and [1], where links to the corresponding literature can be found.

2. Definitions and preliminaries. The graph-theoretical and polyhedral concepts not defined here can be found in Bondy and Murty [3] and Schrijver [14], respectively.

A *graph* G is an ordered pair (V, E) , consisting of a *node set* $V(G)$ and an *edge set* $E(G)$. The edges of G form a subset of $\{\{u, v\} : u, v \in V(G), u \neq v\}$. An edge $\{u, v\}$ is simply denoted by uv . The *complement* of G , denoted \bar{G} , is the graph $(V(G), \{uv : u, v \in V(G), uv \notin E(G)\})$. A *matching* of G is a set of independent edges. A *clique* of G is a subset of mutually adjacent nodes of G . For $v \in V(G)$, $N(v)$ denotes the set $\{v \in V(G) : uv \in E(G)\}$, and for $T \subseteq V(G)$, $N(T)$ denotes the set $\bigcup_{v \in T} N(v) \setminus T$. For a subset F of $V(G)$, the subgraph of G *induced* by F , denoted $G[F]$, is the graph $(F, \{uv \in E(G) : u, v \in F\})$. We will use the following well-known result.

THEOREM 2.1 (Hall's theorem). *If G is a bipartite graph with partition (A, B) , then G has a matching of cardinality $|A|$ if and only if for every $T \subseteq A$, $|N(T)| \geq |T|$.*

Let $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$ and $v \in V(G)$. We denote by $\alpha(G, \mathbf{c})$ the number $\max\{c(S) : S \text{ is a stable set of } G\}$, and $\alpha(G) (= \alpha(G, \mathbf{1}))$ is the *stability number* of G . Any stable set that yields this maximum is referred to as a *maximum weight stable set* of (G, \mathbf{c}) . We use the analogous notion for k -stable sets and the corresponding maximum is denoted by $\alpha_k(G, \mathbf{c})$. Denote by $\delta(v)$ the edges of G that contain v . An *edge covering* (respectively, a *b-matching*) of (G, \mathbf{c}) is collection of edges F such that for every $v \in V(G)$, $|\delta(v) \cap F| \geq c(v)$ (respectively, $|\delta(v) \cap F| \leq c(v)$). In terms of vectors, an edge covering of (G, \mathbf{c}) is any point in

$$\{\mathbf{x} \in \mathbb{Z}_+^{E(G)} : x(\delta(v)) \geq c(v), v \in V(G)\},$$

and a b -matching of (G, \mathbf{c}) is any point in

$$\{\mathbf{x} \in \mathbb{Z}_+^{E(G)} : x(\delta(v)) \leq c(v), v \in V(G)\}.$$

In both case, by letting $\mathbf{x} \in \mathbb{Q}_+^{E(G)}$ we obtain the corresponding notions of *fractional edge coverings* and *fractional b-matchings* of (G, \mathbf{c}) .

Let \mathbf{c} be a vector indexed by $V(G)$, F a subset of $V(G)$. The *restriction of \mathbf{c} to F* is the $|F|$ -dimensional vector whose components correspond to the components of \mathbf{c} indexed by F . We denote by $c(v)$, $v \in V(G)$, the component of \mathbf{c} indexed by v and by $c(F)$ the number $\sum_{v \in F} c(v)$. The *incidence vector* of F , denoted χ^F , is a vector indexed by $V(G)$, with a component equal to one if the corresponding node belongs to F and is equal to zero otherwise. The *support* of \mathbf{c} are those nodes of G that index nonzero components of \mathbf{c} .

To avoid confusion, we digress from the above notation when a vector \mathbf{y} is indexed by a family \mathcal{S} of subsets of $V(G)$. In this case, we use y_S to denote the component of \mathbf{y} indexed by $S \in \mathcal{S}$.

An inequality $\mathbf{c} \cdot \mathbf{x} \leq \delta$ is *implied* by a system of linear inequalities $\mathbf{Ax} \leq \mathbf{b}$ if every \mathbf{x} that satisfies the system also satisfies $\mathbf{c} \cdot \mathbf{x} \leq \delta$.

THEOREM 2.2 (Farkas's lemma). *Suppose that the inequality system $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ has a solution. Then $\mathbf{c} \cdot \mathbf{x} \leq \delta$ is implied by this system if and only if there exists a row vector $\lambda \geq \mathbf{0}$ such that $\lambda \mathbf{A} \geq \mathbf{c}$ and $\lambda \cdot \mathbf{b} \leq \delta$.*

In terms of k -stable sets, Farkas's lemma tells us that $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ defines the k -stable set polytope of a graph G if and only if for every $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$, $\mathbf{c} \cdot \mathbf{x} \leq \alpha_k(G, \mathbf{c})$ is implied by $\mathbf{Ax} \leq \mathbf{b}$. When λ in the above theorem can be chosen to be integral, the system $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ is said to be *totally dual integral*.

Given a polytope Q , an inequality $\mathbf{a} \cdot \mathbf{x} \leq \beta$ is *facet defining* if it is valid for all points of Q and the set $\{\mathbf{x} \in Q : \mathbf{a} \cdot \mathbf{x} = \beta\}$ is a facet of Q .

Padberg [13] has introduced a procedure, called *sequential lifting*, which can be used to build facet defining inequalities for the k -stable set polytope of a graph G from those for induced subgraphs of G . Let $X \subseteq V(G)$, and let $\mathbf{a} \cdot \mathbf{x} \leq \beta$ be a facet defining inequality for the k -stable set of $G[X]$. Let v be any node of $V(G) \setminus X$, and let $\pi = \beta - \max\{\mathbf{a} \cdot \chi^S : S \text{ is a } (k-1)\text{-stable set of } G[X \setminus N(v)]\}$. Then the inequality $\pi x(v) + \mathbf{a} \cdot \mathbf{x} \leq \beta$ is facet defining for the k -stable set polytope of $G[X \cup \{v\}]$. A *lift* of $\mathbf{a} \cdot \mathbf{x} \leq \beta$ (in G) is any inequality obtained by a sequential application of this procedure.

3. Polytopes. In this section we describe the 2-stable set polytope of graphs in general and the 3-stable set of bipartite graphs. We begin with 2-stable sets.

The following theorem includes results of Cook and Shepherd and its proof can be found in [15]. For a subset K of $V(G)$, we denote by $\tilde{N}(K)$ the set $\bigcap_{v \in K} N(v) \setminus K$.

THEOREM 3.1. *For any graph G with $\alpha(G) \leq 2$, the following system describes the stable set polytope of G and is totally dual integral, and each inequality is facet defining.*

- (0) $x(v) \geq 0$ for all $v \in V(G)$.
- (1) $x(K) \leq 1$ for all maximal cliques K .
- (2) $2x(K) + x(\tilde{N}(K)) \leq 2$ for each clique K such that none of the connected components of the complement of $G[\tilde{N}(K)]$ is a bipartite graph.
- (3) $x(V(G)) \leq 2$ if no connected component of the complement of G is bipartite.

Given the above theorem, it is rather easy to describe the 2-stable set of any graph.

DEFINITION 3.2. *For a graph G , the system $\mathbf{A}_2 \mathbf{x} \leq \mathbf{b}$ consists of the inequalities (1)–(3) of Theorem 3.1 as well as the following:*

- (4) $x(A) + 2x(K) + x(\tilde{N}(K) \setminus A) \leq 2$, for each set A such that every maximal stable set in $G[A]$ has size at least 2 and $G[A]$ has a stable set of size at least 3, and for every clique K maximal in $\tilde{N}(A)$.

Note that all inequalities of type (4) can be obtained from the inequality $x(S) \leq 2$, where S is a stable set of size 3, by sequential lifting. If $S = \{v_1, v_2, v_3\}$, the vectors $\chi^{\{v_1, v_2\}}$, $\chi^{\{v_1, v_3\}}$, $\chi^{\{v_2, v_3\}}$ are affinely independent, and they satisfy $x(S) \leq 2$ with equality. So the inequality $x(S) \leq 2$ is facet defining for $G[S]$. When we lift this inequality sequentially to all vertices of a set A that has the property that all stable sets of $G[A]$ have size at least 2, then all vertices of A will have coefficient 1, and we obtain the inequality $x(A) \leq 2$, facet defining for $G[A]$. If we then lift this inequality to a vertex v in $\tilde{N}(A)$, this vertex will get coefficient 2, since the maximal stable set in $G[A \setminus N(v)]$ is of size 0. The same is true for all vertices in a maximal clique K in $\tilde{N}(A)$ that includes v . So we obtain the inequality $x(A) + 2x(K) \leq 2$. When we lift this inequality to the rest of the graph, any vertex v will get coefficient 0 if $K \not\subseteq N(v)$ and coefficient 1 otherwise. So we obtain the inequality $x(A) + 2x(K) + x(\tilde{N}(K) \setminus A) \leq 2$, an inequality of type (4).

Note also that the inequality $x(V(G)) \leq 2$ is implied by the inequality system whenever $\alpha(G) \geq 3$. If no connected component of the complement of G is bipartite, then this inequality is included in type (3). Suppose then that a connected component of the complement of G is bipartite. This means that G contains two cliques, K_1 and K_2 , such that $\tilde{N}(K_1) = \tilde{N}(K_2) = V(G) \setminus (K_1 \cup K_2)$. If $V(G) \setminus (K_1 \cup K_2)$ contains a stable set of size at least 3, then the inequality $x(V) \leq 2$ is implied by two inequalities of type (4), one with $K_1 \subseteq K$, one with $K_2 \subseteq K$, and both with $A \subseteq V(G) \setminus (K_1 \cup K_2)$.

If $V(G) \setminus (K_1 \cup K_2)$ does not contain a stable set of size at least 3, then $\alpha(G) = 2$.

THEOREM 3.3. *For a graph G , the system $\mathbf{A}_2 \mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ describes the 2-stable set polytope of G and is totally dual integral, and each of the inequalities is facet defining.*

Proof. In view of Theorem 3.1, for each inequality of types (1)–(3) with support X , there exists a collection of $|X|$ affinely independent stable sets which satisfy it with equality and which are subsets of X of size at most 2. Therefore, each of these inequalities is facet defining for the 2-stable set polytope of $G[X]$. When the inequalities are lifted to the rest of $V(G)$, they remain facet defining.

For the inequalities of type (4), it is already noted that they can be obtained by lifting the facet defining inequality $x(S) \leq 3$, where S is a stable set of size 3. Therefore, all inequalities of the system $\mathbf{A}_2 \mathbf{x} \leq \mathbf{b}$ are facet defining.

For the rest of the proof, let \mathbf{c} be a nonnegative integral vector indexed by the nodes of G . We show that there exists an integral, nonnegative row vector λ indexed by the rows of \mathbf{A}_2 such that $\lambda \mathbf{A} \geq \mathbf{c}$ and $\lambda \cdot \mathbf{b} \leq \alpha_2(G, \mathbf{c})$.

We apply induction on $\alpha_2(G, \mathbf{c})$. If $\alpha_2(G, \mathbf{c}) = 0$, then $\mathbf{c} = \mathbf{0}$, because every vertex is by itself a stable set. So we can choose $\lambda = \mathbf{0}$. Next, we assume that the theorem holds for any \mathbf{c}' with $\alpha_2(G, \mathbf{c}') < \alpha_2(G, \mathbf{c})$. Furthermore, we may assume without loss of generality that $c(v) > 0$ for all $v \in V(G)$, since nodes with zero weight can be deleted from the graph. (This is because any of the above inequalities can be lifted to an inequality in G that contains v and is of one of the specified types.)

Note that it suffices to find an inequality $\mathbf{a} \cdot \mathbf{x} \leq \beta$ of the system $\mathbf{A}_2 \mathbf{x} \leq \mathbf{b}$ such that $\alpha_2(G, \mathbf{c} - \mathbf{a}) \leq \alpha_2(G, \mathbf{c}) - \beta$ and $\mathbf{c} - \mathbf{a} \geq \mathbf{0}$. Then, by induction, there exists a nonnegative integral vector λ' such that $\lambda' \mathbf{A}_2 \geq \mathbf{c} - \mathbf{a}$ and $\lambda' \cdot \mathbf{b} \leq \alpha_2(G, \mathbf{c} - \mathbf{a})$. Let λ be obtained from λ' by simply increasing the coordinate of λ' indexed by this inequality by 1. This yields $\lambda \mathbf{A}_2 \geq \mathbf{c}$ and $\lambda \cdot \mathbf{b} \leq \alpha_2(G, \mathbf{c} - \mathbf{a}) + \beta \leq \alpha_2(G, \mathbf{c})$.

If $\alpha(G) \leq 2$, then we are done by Theorem 3.1. Thus we assume that $\alpha(G) \geq 3$. If every maximum weight 2-stable set has cardinality 2, then let $\mathbf{a} \cdot \mathbf{x} \leq \beta$ be the inequality $x(V(G)) \leq 2$, if either this inequality or the weighted sum of the two inequalities of type (4) that implies the inequality $x(V(G)) \leq 2$ are included in the system (see the note following the statement of the Theorem). In this case it is immediate that $\alpha_2(G, \mathbf{c} - \mathbf{a}) \leq \alpha_2(G, \mathbf{c}) - \beta$. Otherwise, there exists a node v such that $c(v) = \alpha_2(G, \mathbf{c})$ (and $\alpha_2(G, \mathbf{c}) \geq 2$), and thus $N(v) = V(G) \setminus \{v\}$. Let K be a clique of G that contains v such that $\tilde{N}(K) = V(G) \setminus K$ and K is maximal with this property. Let $\mathbf{a} \cdot \mathbf{x} \leq \beta$ be the inequality $2x(K) + x(\tilde{N}(K)) \leq 2$. Because of the maximality condition on K , all stable sets in $G[V(G) \setminus K]$ have size at least 2, so this inequality is of type (4), with $A = V(G) \setminus K$. Note that every maximum weight 2-stable set of G consists of either an element of K or two elements of $\tilde{N}(K)$. Thus, since $c(v) \geq 2$ and because K must include all vertices v such that $c(v) = \alpha_2(G, \mathbf{c})$, $\alpha_2(G, \mathbf{c} - \mathbf{a}) \leq \alpha_2(G, \mathbf{c}) - \beta$. The proof is complete. \square

Before we can adequately describe the facet defining inequalities of the 3-stable set polytope of bipartite graphs, we need a few definitions.

DEFINITION 3.4. *A 2-star is a graph whose node set can be partitioned into two sets $T = \{v_1, v_2\}$ and B such that T is a stable set and $B = N(T)$. With any given 2-star we identify the sets $L = N(v_1) \setminus N(v_2)$, $M = N(v_1) \cap N(v_2)$, and $R = N(v_2) \setminus N(v_1)$. A 2-star is full if $|N(v_1)|, |N(v_2)| \geq 3$ and $|L|, |R| \geq 2$.*

DEFINITION 3.5. *For a bipartite graph G , the inequality system $\mathbf{A}_3 \mathbf{x} \leq \mathbf{b}$ consists of the following inequalities:*

- (1) $x(K) \leq 1$ for all maximal cliques K of G (i.e., edges and isolated nodes);

- (2) $x(V(G)) \leq 3$ if both parts of a bipartition of G have at least four elements;
 (3) the lift of $x(S) \leq 3$ for each stable set S of size at least 4;
 (4) $2x(T) + 2x(u) + x(B \setminus \{u\}) \leq 4$ for each full 2-star with $M \neq \emptyset$ and for each node $u \in M$;
 (5) $4x(v_1) + 2x(v_2) + 2x(N(v_1)) + x(R) \leq 6$ for each full 2-star with $|R| \geq 3$ and $|N(v_1)| \geq 4$ if $M \neq \emptyset$ and each ordering (v_1, v_2) of T .

When the inequality $x(S) \leq 3$, where S is a stable set of size at least 4, is lifted to the rest of the graph, and a vertex v is encountered that is adjacent to all previously lifted vertices, then v will get coefficient 3 in the inequality. All other vertices will then have coefficient 1 if they are adjacent to v , and 0 if this is not the case. If a vertex v is adjacent to all previously lifted vertices except one single vertex or two adjacent vertices, then v will get coefficient 2. All other vertices will get coefficient 1 if they are adjacent to v , or 0 if this is not the case. Otherwise, the inequality will be lifted to $x(V(G)) \leq 3$. So the type (3) inequalities are of the following form:

- (3a) $3x(v) + x(N(v)) \leq 3$ for all $v \in V(G)$ with $|N(v)| \geq 4$;
 (3b) $2x(v) + x(N(v)) + x(K) \leq 3$ for all $v \in V(G)$ and each clique K with no element in common with $N(v) \cup \{v\}$;
 (3c) $x(V(G)) \leq 3$ if G cannot be obtained from $K_{m,n}$, $m \geq 1$, $n \geq 4$, by deleting a (possibly empty) matching.

THEOREM 3.6. *For a bipartite graph G , the normalized version of the system $\mathbf{A}_3 \mathbf{x} \leq \mathbf{b}$ (i.e., where all inequalities have been divided by the right-hand side), $\mathbf{x} \geq \mathbf{0}$ is totally dual integral and describes the 3-stable set polytope of G , and each inequality is facet defining.*

Proof. We begin by showing that each of the inequalities given is facet defining. This is done by exhibiting, in each case, the appropriate sets of affinely independent vectors that satisfy the given inequality with equality. For convenience, we will say that we find affinely independent stable sets, instead of affinely independent incidence vectors of stable sets. The following fact can be proved by using the notion of lifting as in the proof of Theorem 3.3.

STEP 3.6.1. *If S , $|S| \geq 4$ (respectively, $|S| \geq 3$), is a stable set of G , then it has $|S|$ affinely independent subsets of size 3 (respectively, 2).*

The inequalities $\mathbf{x} \geq \mathbf{0}$ and those of type (1) are well known to be facet defining. From (3.6.1) and the lifting procedure, it is immediate that the inequalities of type (3) are facet defining. The same lemma may also be applied to each part of a bipartition of G and thus the inequality of type (2) is facet defining as well. Consider now an inequality of type (4). By (3.6.1), $B \setminus \{u\}$ has $|B| - 1$ affinely independent stable sets of two nodes and thus, by including u to all these sets, B has $|B| - 1$ such sets of three nodes. Adding also the sets $\{v_1, v_2\}$, $\{v_1, u_3, u_4\}$, and $\{v_2, u_1, u_2\}$, where $u_1, u_2 \in L$ and $u_3, u_4 \in R$, we obtain a collection of $|B| + |T|$ affinely independent 3-stable sets that satisfy the given inequality with equality.

Finally, consider an inequality of type (5), and assume that $|N(v_1)| \geq 4$. $N(v_1)$ contains $|N(v_1)|$ affinely independent stable sets of size 3. Also, R has $|R|$ affinely independent stable sets of size 2. By including v_1 to these sets, we have that $R \cup \{v_1\}$ has $|R|$ affinely independent stable sets of size 3. Adding the sets $\{v_1, v_2\}$ and $\{v_2, u_1, u_2\}$, where $u_1, u_2 \in L$, we obtain the required collection of affinely independent stable sets. When $|N(v_1)| = 3$ instead of $N(v_1)$, we use $L \cup \{v_2\}$ to obtain $|L|$ affinely independent 3-stable sets.

We continue with the rest of the proof. Let \mathbf{c} be a nonnegative integral vector indexed by the nodes of G . We show that there exists an integral, nonnegative row

vector λ indexed by the rows of \mathbf{A}_3 such that $\lambda \mathbf{A}'_3 \geq \mathbf{c}$ and $\lambda \cdot \mathbf{1} \leq \alpha_3(G, \mathbf{c})$, where $\mathbf{A}'_3 \mathbf{x} \leq \mathbf{1}$ is the normalized version of the system $\mathbf{A}_3 \mathbf{x} \leq \mathbf{b}$.

The proof is by induction on $\alpha_3(G, \mathbf{c})$. If $\alpha_3(G, \mathbf{c}) = 0$, then $\mathbf{c} = \mathbf{0}$, so we can take $\lambda = \mathbf{0}$. Next, we assume that the lemma holds for any \mathbf{c}' with $\alpha_3(G, \mathbf{c}') < \alpha_3(G, \mathbf{c})$. Furthermore, we may assume without loss of generality that $c(v) > 0$ for all $v \in V(G)$. (This is because any of the above inequalities for $G - v$, $v \in V(G)$, can be lifted to an inequality of G that contains v and is of one of the specified types.) As in the proof of Theorem 3.3, it suffices to find an inequality $\mathbf{a} \cdot \mathbf{x} \leq \beta$ which is the sum (with the appropriate coefficients) of inequalities of $\mathbf{A}_3 \mathbf{x} \leq \mathbf{b}$ and such that $\alpha_3(G, \mathbf{c} - \mathbf{a}) \leq \alpha_3(G, \mathbf{c}) - \beta$ and $\mathbf{c} - \mathbf{a} \geq \mathbf{0}$.

If every stable set of G is of size at most 3, then the stable set and 3-stable set polytopes of G coincide, and it is well known that the former is given by the inequalities of types (0) and (1). Otherwise, if every maximum weight 3-stable set of G is of size exactly 3 and for all $v \in V(G)$, $c(v) < \alpha_3(G, \mathbf{c}) - 1$, then the required inequality is $x(V(G)) \leq 3$. Clearly, $\alpha_3(G, \mathbf{c} - \chi^{V(G)}) = \alpha_3(G, \mathbf{c}) - 3$. If the inequality $x(V(G)) \leq 3$ is not of type (2) or (3c), then by the definition of $\mathbf{A}_3 \mathbf{x} \leq \mathbf{b}$, each bipartition of G has a part with less than four nodes, and G can be obtained from $K_{m,n}$, $m \geq 1$, $n \geq 4$ by deleting a matching. If G can be obtained from $K_{3,n}$, $n \geq 4$, by deleting a matching, then $x(V(G)) \leq 3$ is the sum (with coefficients $\frac{1}{3}$) of three inequalities of type (3b). If G can be obtained from $K_{2,n}$ or $K_{1,n}$ by deleting a matching, then $x(V(G))$ is implied by an inequality of type (3b) or (3a).

Thus we may assume that G has a maximal 3-stable set T of size 1 or 2 and if $|T| = 2$, then T is of maximum weight. Let $B = N(T)$. By increasing the weight of components of \mathbf{c} that correspond to vertices that are not part of maximum weight stable sets, if necessary, we may thus assume the following.

STEP 3.6.2. *Every node of G belongs to a maximum weight 3-stable set of (G, \mathbf{c}) .*

We now distinguish two cases.

Case 1. G is not a full 2-star.

If T has only one element v , then every 3-stable set of maximum weight consists of either v or three nodes in $N(v)$. Thus $3x(v) + x(N(v)) \leq 3$ is the required inequality, since by (3.6.2), $c(v) = \alpha_3(G, \mathbf{c}) \geq 3$.

Thus we may assume that G is a 2-star, although not a full one, and since $\alpha_3(G) > 3$, $|B| \geq 4$. Let v_1 and v_2 be the elements of T , ordered so that $|N(v_2)| \leq |N(v_1)|$. It can be argued from the fact that $c(v_1) + c(v_2) = \alpha_3(G, \mathbf{c})$ that $c(v_1) \geq 2$. Let $\mathbf{a} \cdot \mathbf{x} \leq \beta$ be the inequality $2x(v_1) + x(N(v_1)) + x(K) \leq 3$ (of type (3b)), where K is a maximal clique in $\{v_2\} \cup R$. Every maximum weight 3-stable set of (G, \mathbf{c}) is also of maximum weight in (G, \mathbf{a}) and $\mathbf{c} - \mathbf{a} \geq \mathbf{0}$. Thus $\alpha_3(G, \mathbf{c} - \mathbf{a}) \leq \alpha_3(G, \mathbf{c}) - \beta$, as required.

Case 2. G is a full 2-star.

We adopt the notation introduced in Definition 3.4. To proceed, we need to identify certain nodes of G and introduce a simple lemma. Let t_1 and t_2 be the weights of v_1 and v_2 , and let m_1, m_2, \dots be the weights of the nodes of M , ℓ_1, ℓ_2, \dots the weights of the nodes of L , and r_1, r_2, \dots the weights of the nodes of R , all in decreasing order.

STEP 3.6.3. *There is no 3-stable set S of maximum weight such that $S \cap L \neq \emptyset$, $S \cap R \neq \emptyset$, and $S \cap M = \emptyset$.*

This can be deduced from the fact that $r_1 + r_2 \leq t_2$ and $\ell_1 + \ell_2 \leq t_1$ (because $t_1 + t_2 = \alpha_3(G, \mathbf{c})$), and thus both $\ell_1 + r_1 + r_2$ and $\ell_1 + \ell_2 + r_1$ are less than $t_1 + t_2$.

We consider three subcases.

The simplest case arises when no maximum weight 3-stable set of G is a subset

of B . Let $\mathbf{a} \cdot \mathbf{x} \leq \beta$ denote the inequality $2x(T) + x(B) \leq 4$. Since $t_1 + t_2 = \alpha_2(G, \mathbf{c})$, $t_1 \geq \ell_1 + \ell_2 \geq 2$, $t_2 \geq r_1 + r_2 \geq 2$ and thus $\mathbf{c} - \mathbf{a} \geq \mathbf{0}$. In addition, $\alpha_3(G, \mathbf{c} - \mathbf{a}) \leq \alpha_3(G, \mathbf{c}) - \beta$. Now if $M \neq \emptyset$, then $\mathbf{a} \cdot \mathbf{x} \leq \beta$ is implied by an inequality of type (4); if $M = \emptyset$, then it is implied by the two inequalities of type (5) (with coefficients $\frac{1}{3}$) that correspond to the two orderings of T .

Next, suppose there is a 3-stable set of maximum weight that contains nodes from each of L , R , and M —in other words, $r_1 + \ell_1 + m_1 = t_1 + t_2$. Our analysis depends on whether or not $|M| \geq 2$ and $m_1 = m_2$.

If $|M| \geq 2$ and $m_1 = m_2$, then let $\mathbf{a} \cdot \mathbf{x} \leq \beta$ be the inequality $3x(T) + x(B) + x(\{u_\ell, u_r\})$, where u_ℓ and u_r are the nodes of weight ℓ_1 and r_1 , respectively. Now by (3.6.3), $r_1 + \ell_1 + \ell_2 < \alpha_3(G, \mathbf{c}) = \ell_1 + m_1 + r_1$, so $\ell_2 < m_1$, and thus $\ell_1 + m_1 + m_2 > \ell_1 + \ell_2 + m_1$. By symmetry, $r_1 + m_1 + m_2 > r_1 + r_2 + m_2$. Therefore, u_ℓ and u_r are the only nodes from L and R , respectively, that are contained in maximum weight stable sets. Also, since $\ell_1 > \ell_2 \geq 1$ and $r_1 > r_2 \geq 1$, $\ell_1, r_1 \geq 2$, and since $\ell_1 + \ell_2 \leq t_1$ and $r_1 + r_2 \leq t_2$, we have that $t_1, t_2 \geq 3$. It follows that $\mathbf{a} \cdot \mathbf{x} \leq \beta$ has the property that all the maximum weight 3-stable sets of (G, \mathbf{c}) are also maximum weight stable sets of (G, \mathbf{a}) , and thus $\alpha_3(G, \mathbf{c} - \mathbf{a}) \leq \alpha_3(G, \mathbf{c}) - \beta$, since $\mathbf{c} - \mathbf{a} \geq \mathbf{0}$. In addition, $\mathbf{a} \cdot \mathbf{x} \leq \beta$ is the sum of two inequalities of type (3b).

If $|M| = 1$ or $m_1 > m_2$, then all maximum weight 3-stable sets in B have to include the node $u \in M$ of weight m_1 . Also, since $\ell_1 + \ell_2 \leq t_2$, $r_1 + r_2 \leq t_1$, and because, by (3.6.3), $r_1 + r_2 + \ell_1 \leq \alpha_3(G, \mathbf{c}) = r_1 + m_1 + \ell_1$ (so $m_1 > r_2$), we have that $t_1, t_2, m_1 \geq 2$. Thus by taking $\mathbf{a} \cdot \mathbf{x} \leq \beta$ to be the type (4) inequality $2x(T) + 2x(u) + x(B \setminus \{u\}) \leq 4$, we have $\mathbf{c} - \mathbf{a} \geq \mathbf{0}$ and $\alpha_3(G, \mathbf{c} - \mathbf{a}) \leq \alpha_3(G, \mathbf{c}) - \beta$.

To conclude, we assume that every maximum weight 3-stable set that is contained in B is also contained in, say, $N(v_1)$. The inequality $\mathbf{a} \cdot \mathbf{x} \leq \beta$ of the form $4x(v_1) + 2x(v_2) + 2x(N(v_1)) + x(R) \leq 6$ has the property that every maximum weight stable set of (G, \mathbf{c}) is also a maximum weight stable set of (G, \mathbf{a}) . If $|R| = 2$, this inequality is the sum of two inequalities of type (3b); if $M \neq \emptyset$ and $|N(v_1)| = 3$, it is the sum of two inequalities of type (1) and one inequality of type (4); otherwise it is an inequality of type (5). In either case we are done, provided that $t_1 \geq 4$, $t_2 \geq 2$, and $c(u) \geq 2$ for all $u \in N(v_1)$.

Since $t_1 + t_2 = \alpha_3(G, \mathbf{c})$, $t_1 + t_2 \geq t_1 + r_1 + r_2$ and thus $t_2 \geq 2$. Next we show that $c(u) \geq 2$ for all $u \in N(v_1)$. If $u \in M$, then by (3.6.2), there is a 3-stable set $S = \{u, w_1, w_2\}$ of maximum weight that contains u . Since $u \in M$, $S \subseteq B$. By the assumption that every maximum weight stable set is contained in $N(v_1)$, $c(w_1) + c(w_2) + r_1 < \alpha_3(G, \mathbf{c}) = c(u) + c(w_1) + c(w_2)$. Thus $c(u) \geq 2$. Now if $u \in L$ we may assume that $c(u)$ is minimal. Suppose that every maximum weight stable set that contains u also contains v_2 . (If this is not so, we are done by the same reasoning as for $u \in M$.) Then $c(u) \geq \ell_2$, and since $c(u)$ is minimal, $c(u) = \ell_2$ and $\ell_1 + 2\ell_2 < \alpha_3(G, \mathbf{c})$. By assumption, there is a maximum weight stable set S' contained in B , but by the above, it does not contain a node with weight ℓ_2 . Thus $|M| \geq 2$ and $m_1 + m_2 + \ell_1 \leq \alpha_3(G, \mathbf{c}) = \ell_1 + \ell_2 + t_2$. But, by (3.6.2), $r_1 + r_2 = t_2 \geq m_1 + m_2 - \ell_2$ and thus $\ell_2 \geq m_1 + m_2 - r_1 - r_2$. But we saw that $m_i > r_1$ (so $m_i - r_1 \geq 1$) for all i , so $\ell_2 \geq 2$. Thus for all $u \in N(v_1)$, $c(u) \geq 2$. Finally, since $t_1 + t_2 = \alpha_3(G, \mathbf{c}) \geq t_2 + \ell_1 + \ell_2 \geq t_2 + 4$, we have $t_1 \geq 4$, as required. \square

Parenthetically, we remark that the ideas behind the inequalities of types (4) and (5) of Definition 3.5 can be extended to construct facet defining inequalities for the stable set polytopes of graphs. Let H be a graph such that $x(V(H)) \leq \alpha(H)$ is a facet defining inequality for the stable set polytope of H and H has two stable sets S_1, S_2

with $|S_1| = |S_2| = \alpha(H)$ and $S_1 \cap S_2 = \emptyset$. Let G be a graph obtained from H by adding three new nodes v_1, v_2, u and for $i = 1, 2$ the edges $\{v_i u\} \cup \{v_i v : v \in V(H) \setminus S_i\}$. Let $\alpha = \alpha(G)$. Then the inequality

$$(\alpha - 1)x(\{v_1, v_2, u\}) + x(V(H)) \leq 2\alpha - 2$$

is facet defining for the stable set polytope of G . Alternatively, let H_1 and H_2 be two graphs such that the inequalities $x(V(H_1)) \leq \alpha(H_1)$ and $x(V(H_2)) \leq \alpha(H_2)$ are facet defining for the stable set polytope of H_1 and H_2 , respectively. Let G be obtained from H_1 and H_2 by adding two new nodes v_1 and v_2 and the edges $\cup_{i=1}^2 \{v_i v : v \in V(H_i)\} \cup \{v_1 v_2\}$. Let $\alpha = \alpha(G)$. Then the inequality

$$(\alpha(\alpha - 2) + 1)x(v_1) + (\alpha - 1)x(v_2) + (\alpha - 1)x(V(H_1)) + x(V(H_2)) \leq \alpha(\alpha - 1)$$

is facet defining for the stable set polytope of G . The proof in both cases is analogous to the one given in Theorem 3.6. (Examples illustrating both constructions are shown in Figure 3.1; the coefficients not shown are equal to 1 and the right-hand sides are, respectively, 6 and 12; in both cases $\alpha = 4$.) We note that the above constructions can be generalized to obtain many families of facet defining inequalities for the stable set polytopes of graphs, but this falls beyond the scope of the present paper.

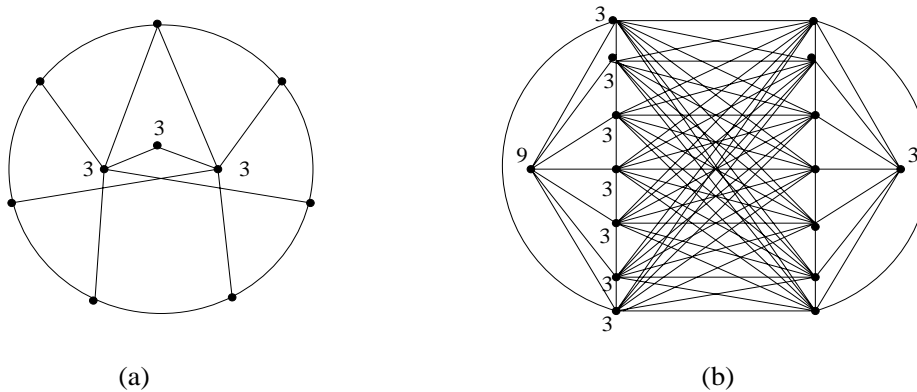


FIG. 3.1. An illustration of facet defining inequalities for the stable set polytope.

4. Colorings. We will now use the results of the previous section to obtain upper bounds on the value of integral colorings of the corresponding cases involved.

Note first that a description of the k -stable set polytope of a graph G provides necessary and sufficient conditions on the fractional colorability of (G, \mathbf{c}) by k -stable sets, where \mathbf{c} is any vector in $\mathbb{Z}_+^{V(G)}$. Indeed, by definition, \mathbf{y} is a fractional coloring of (G, \mathbf{c}) of value r if and only if $\frac{1}{r}\mathbf{c}$ belongs to the k -stable set of G , i.e., if and only if $\mathbf{A}_k \mathbf{c} \leq r\mathbf{b}$, where $\mathbf{A}_k \mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ is an inequality description of the k -stable set of G . Thus,

$$\eta_k(G, \mathbf{c}) = \max\left\{\frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c}) : \mathbf{a} \cdot \mathbf{x} \leq \beta \text{ is an inequality of } \mathbf{A}_k \mathbf{x} \leq \mathbf{b}\right\}.$$

We will use this min-max equality throughout. To begin, we introduce a simple lemma that explains how color classes from optimal fractional colorings intersect with the support of any given valid inequality of the k -stable set polytope of a graph.

LEMMA 4.1. *Let G be a graph and $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$. Let $\mathbf{a} \cdot \mathbf{x} \leq \beta$, $\beta \neq 0$, (\mathbf{a}, β) integral and nonnegative, be a valid inequality for the k -stable set polytope of G . If $\mathbf{a} \cdot \mathbf{c} = (\eta_k(G, \mathbf{c}) - t)\beta$, then in any optimal fractional coloring \mathbf{y} of (G, \mathbf{c}) by k -stable sets, $\sum_{\{S \in \mathcal{S}: a(S) < \beta\}} y_S \leq t\beta$.*

Proof. Let \mathcal{S}_0 (respectively, \mathcal{S}_1) consist of those k -stable sets $S \in \mathcal{S}$ such that $a(S) = \beta$ (respectively, $a(S) \leq \beta - 1$). Let $p_0 = \sum_{S \in \mathcal{S}_0} y_S$ and $p_1 = \sum_{S \in \mathcal{S}_1} y_S$. Since $\sum_{S \in \mathcal{S}_0} y_S \chi^S + \sum_{S \in \mathcal{S}_1} y_S \chi^S \geq \mathbf{c}$,

$$\begin{aligned} \beta(\eta_k(G, \mathbf{c}) - t) &= \mathbf{a} \cdot \mathbf{c} \\ &\leq \mathbf{a} \cdot (\sum_{S \in \mathcal{S}_0} y_S \chi^S + \sum_{S \in \mathcal{S}_1} y_S \chi^S) \\ &= \sum_{S \in \mathcal{S}_0} y_S a(S) + \sum_{S \in \mathcal{S}_1} y_S a(S) \\ &\leq p_0 \beta + p_1 (\beta - 1) \\ &= \eta_k(G, \mathbf{c}) - p_1, \end{aligned}$$

as required. \square

Next we show that an upper bound on colorings by 2-stable sets can be obtained as the solution of two linear programs.

THEOREM 4.2. *For any graph G and any (strictly) positive vector $\mathbf{c} \in \mathbb{Z}^{V(G)}$, $\chi_2(G, \mathbf{c}) \leq \eta_2(G, \mathbf{c}) + \frac{\eta(G)}{2}$.*

Proof. We proceed by induction on $\eta_2(G, \mathbf{c})$. That is, we assume that the theorem holds for any graph G' and $\mathbf{c}' \in \mathbb{Z}_+^{V(G')}$ with $\eta_2(G', \mathbf{c}') \leq \eta_2(G, \mathbf{c}) - 1$. In the base case, G is the empty graph and the theorem holds trivially.

If the only inequality $\mathbf{a} \cdot \mathbf{x} \leq \beta$ of $\mathbf{A}_2 \mathbf{x} \leq \mathbf{b}$ for which $\mathbf{a} \cdot \mathbf{c} > (\eta_2(G, \mathbf{c}) - 1)\beta$ is the inequality $x(V(G)) \leq 2$ (of type (3)), then let S be any 2-stable set of G of size 2. Let $\mathbf{d} = \mathbf{c} - \chi^S$. Then $\mathbf{A}_2 \mathbf{d} \leq (\eta_2(G, \mathbf{c}) - 1)\mathbf{b}$, and consequently $\eta_2(G, \mathbf{d}) \leq \eta_2(G, \mathbf{c}) - 1$. Let $G' = G - \{v \in V(G) : d(v) = 0\}$ and let \mathbf{c}' be the restriction of \mathbf{d} to $V(G')$. By induction, $\chi_2(G', \mathbf{c}') \leq \eta_2(G', \mathbf{c}') + \frac{\eta(G')}{2}$. Since any coloring of (G, \mathbf{c}') can be augmented with S to form a coloring of (G, \mathbf{c}) ,

$$\chi_2(G, \mathbf{c}) \leq \eta_2(G', \mathbf{c}') + \frac{\eta(G')}{2} + 1 \leq \eta_2(G, \mathbf{c}) + \frac{\eta(G)}{2},$$

as required.

Thus we assume that there is an inequality $\mathbf{a} \cdot \mathbf{x} \leq \beta$ of $\mathbf{A}_2 \mathbf{x} \leq \mathbf{b}$ that is not of type (3) such that $\mathbf{a} \cdot \mathbf{c} > \eta_2(G, \mathbf{c}) - 1$. Choose this inequality so that for any other inequality $\mathbf{a}' \cdot \mathbf{x} \leq \beta'$ that is not of type (3), $\frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c}) \geq \frac{1}{\beta'}(\mathbf{a}' \cdot \mathbf{c})$. Let K be the clique involved in the definition of the support of \mathbf{a} . If the support of \mathbf{a} is K , let G' be the empty graph and G'' be G itself. Otherwise, let G' be the subgraph $G[\tilde{N}(K)]$ of G and G'' the subgraph $G - \tilde{N}(K)$ of G . Let \mathbf{c}' be the restriction of \mathbf{c} to $V(G')$, and let \mathbf{c}'' be the restriction of \mathbf{c} to $V(G'')$. We now use induction to color (G', \mathbf{c}') , whereas we color (G'', \mathbf{c}'') explicitly.

Consider (G', \mathbf{c}') and assume that G' is nonempty. By definition of the support of \mathbf{a} , every stable set in the graph $G[V(G') \cup K]$ that contains an element of K is of size 1. Also, by the choice of $\mathbf{a} \cdot \mathbf{x} \leq \beta$, $G[V(G') \cup K]$ with \mathbf{c} restricted to $V(G') \cup K$ has a fractional coloring by 2-stable sets of value $\frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c})$. If we remove from this coloring all stable sets of size 1 that consist of a node of K , we are left with a fractional coloring of G' of value $\frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c}) - c(K)$. Thus $\eta_2(G', \mathbf{c}') \leq \frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c}) - c(K)$. It follows, by induction, that

$$\chi_2(G', \mathbf{c}') \leq \eta_2(G', \mathbf{c}') + \frac{\eta(G')}{2} \leq \frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c}) - c(K) + \frac{\eta(G')}{2}.$$

Consider (G'', \mathbf{c}'') . Let P be the node set of $G - (V(G') \cup K)$. Let $t = \eta_2(G, \mathbf{c}) - \frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c})$. Note that $t = 0$ or $t = \frac{1}{2}$. We show that $\chi_2(G'', \mathbf{c}'') \leq c(K) + 2t$.

We construct a bipartite graph H with node set $A \cup B$ as follows. For each $v \in K$ (respectively, $v \in P$), let A_v (respectively, B_v) be a set of $c(v)$ nodes. Let $A = \cup_{v \in K} A_v$, $B = \cup_{v \in P} B_v$. To define the edges of H , consider an optimal fractional coloring of (G, \mathbf{c}) by 2-stable sets and join all elements of A_u to all elements of B_v if and only if there is a color class that consists of u and v . If $|A| > |B|$, we expand B with $|A| - |B|$ additional nodes, each joined to all nodes of A . Note that the way H is constructed guarantees that $|N(T)| \geq |T|$ for each $T \subseteq A$. Thus, by Theorem 2.1, H has a matching of size $|A|$. Also, at most $2t$ nodes of B are not contained in this matching. This is because, by Lemma 4.1, there are at most weight $2t$ color classes S of the fractional coloring that do not contain an element of K and $S \cap (V(G) \setminus P) \leq 1$. Now in an obvious manner, this matching and the node not saturated by it, if it exists, correspond to a coloring of (G'', \mathbf{c}'') with at most $c(K) + 2t$ 2-stable sets. Thus, $\chi_2(G'', \mathbf{c}'') \leq c(K) + 2t$.

The proof is now complete:

$$\begin{aligned} \chi_2(G, \mathbf{c}) &\leq \chi_2(G'', \mathbf{c}'') + \chi_2(G', \mathbf{c}') \leq c(K) + 2t + \frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c}) - c(K) + \frac{\eta(G')}{2} \\ &\leq \eta_2(G, \mathbf{c}) + \frac{\eta(G)-1}{2} + t \leq \eta_2(G, \mathbf{c}) + \frac{\eta(G)}{2}, \end{aligned}$$

since $\eta(G') \leq \eta(G) - 1$, $\frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c}) + t = \eta_2(G, \mathbf{c})$, and $t \leq \frac{1}{2}$. \square

Note that in the above proof, if $c(V(G)) < 2\eta_2(G, \mathbf{c})$, then there must be an inequality that is not of type (3) for which $\frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c}) = \eta_2(G, \mathbf{c})$. Thus we are in the second case and $t = 0$. In this case the proof has demonstrated the following, slightly stronger, statement. We will use it in the next theorem.

COROLLARY 4.3. *Let G be a graph, $\mathbf{c} \in \mathbb{Z}^{V(G)}$, $\mathbf{c} > \mathbf{0}$, and $r \geq \eta_2(G, \mathbf{c})$. If $c(V(G)) < 2r$, then $\chi_2(G, \mathbf{c}) < r + \frac{\eta(G)}{2}$.*

The upper bound of Theorem 4.2 is tight. If, for instance, G is the complete p -partite graph with each color class having an odd number of nodes, then $\chi_2(G) = \eta_2(G) + \frac{\eta(G)}{2}$.

We note that for any graph G and any vector $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$, $\chi_2(G, \mathbf{c})$ can be computed efficiently using the theory of matchings. First observe that any $v \in V(G)$ such that $N(v) \cup \{v\} = V(G)$ will account for $c(v)$ color classes in any optimal coloring of (G, \mathbf{c}) by 2-stable sets. Thus, we can ignore them and assume that every node of G belongs to a stable set of size 2. As a consequence, we may further assume that in any optimal coloring of (G, \mathbf{c}) by 2-stable sets, all color classes are of cardinality 2. Thus optimal colorings of (G, \mathbf{c}) correspond to optimal edge coverings of (\bar{G}, \mathbf{c}) . Now in turn, it is well known that optimal edge coverings of (\bar{G}, \mathbf{c}) can be computed with the aid of optimal b -matchings of (G, \mathbf{c}) . Namely, given an optimal b -matching \mathbf{x} of (\bar{G}, \mathbf{c}) , greedily construct a vector $\mathbf{x}' \in \mathbb{Z}_+^{E(\bar{G})}$ such that $\mathbf{x} + \mathbf{x}'$ is an edge covering of (\bar{G}, \mathbf{c}) . (Note that $\mathbf{1} \cdot (\mathbf{x} + \mathbf{x}') = \mathbf{1} \cdot \mathbf{c}$.) It is straightforward to verify that $\mathbf{x} + \mathbf{x}'$ is an optimal edge cover of (\bar{G}, \mathbf{c}) . We note finally that the problem of finding an optimal b -matching in (\bar{G}, \mathbf{c}) can be solved efficiently using Edmonds's matching polyhedron theorem [6].

We would like to mention that the relationship between b -matchings and edge coverings outlined here is based on Gallai's theorem (see [3]) concerning the case where \mathbf{c} is the all-ones vector. Also, this relationship can be extended to the fractional counterparts of edge coverings and b -matchings. In this case, for any graph G and

any positive $\mathbf{c} \in \mathbb{Z}^{V(G)}$ the inequality of Theorem 4.2 implies that

$$\mu(G, \mathbf{c}) \geq \mu'(G, \mathbf{c}) - \frac{\eta(H)}{2},$$

where $\mu(G, \mathbf{c})$ and $\mu'(G, \mathbf{c})$ are the values of an optimal b -matching and an optimal fractional b -matching of (G, \mathbf{c}) , respectively, and $H = \bar{G} - \{v : N(v) = \emptyset\}$.

We now turn our attention to colorings by 3-stable sets.

THEOREM 4.4. *For any bipartite graph G and $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$, $\chi_3(G, \mathbf{c}) \leq \lceil \eta_3(G, \mathbf{c}) \rceil + 1$.*

The crux of the proof is the following result.

LEMMA 4.5. *Let G be a bipartite graph and $\mathbf{c} \in \mathbb{Z}_+^{V(G)}$. If one of the following two conditions holds, then $\chi_3(G, \mathbf{c}) = \lceil \eta_3(G, \mathbf{c}) \rceil$.*

(i) $c(v) < \eta_3(G, \mathbf{c})$ and $\frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c}) = \eta_3(G, \mathbf{c})$ for an inequality $\mathbf{a} \cdot \mathbf{x} \leq \beta$ of $\mathbf{A}_3\mathbf{x} \leq \mathbf{b}$, which is not of type (1).

(ii) G can be obtained from a 2-star by deleting zero or more nodes.

Proof. Any coloring is also a fractional coloring and thus $\chi_3(G, \mathbf{c}) \geq \lceil \eta_3(G, \mathbf{c}) \rceil$. We show the reverse inequality by induction on $\eta_3(G, \mathbf{c})$. That is, we assume that the lemma holds for any $\mathbf{c}' \in \mathbb{Z}_+^{V(G)}$ such that $\eta_3(G, \mathbf{c}') \leq \lceil \eta_3(G, \mathbf{c}) \rceil - 1$. With no loss of generality, $c(v) > 0$ for all $v \in V(G)$. Denote by \mathcal{S} the color classes of an optimal fractional coloring \mathbf{y} of (G, \mathbf{c}) . We distinguish two cases.

Case 1. (G, \mathbf{c}) fulfills condition (ii).

Suppose that G is a 2-star. With the notation of Definition 3.4, we assume that $|R| \geq |L|$. Let S be a member of \mathcal{S} that contains v_1 . If necessary, include additional nodes in S so that it is either maximal or $|S| = 3$. Let $\mathbf{a} \cdot \mathbf{x} \leq \beta$ be an inequality of $\mathbf{A}_3\mathbf{x} \leq \mathbf{b}$. If $\mathbf{a} \cdot \mathbf{c} = \beta\eta_3(G, \mathbf{c})$, then by Lemma 4.1, $\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \chi^S = \beta(\eta_3(G, \mathbf{c}) - 1)$. Otherwise, it can be readily checked that $\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \chi^S \leq \beta(\lceil \eta_3(G, \mathbf{c}) \rceil - 1)$. Thus $\eta_3(G, \mathbf{c} - \chi^S) \leq \lceil \eta_3(G, \mathbf{c}) \rceil - 1$ and, by induction, $\chi_3(G, \mathbf{c} - \chi^S) \leq \lceil \eta_3(G, \mathbf{c} - \chi^S) \rceil$. It follows that

$$\chi_3(G, \mathbf{c}) \leq \chi_3(G, \mathbf{c} - \chi^S) + 1 \leq \lceil \eta_3(G, \mathbf{c} - \chi^S) \rceil + 1 \leq \lceil \eta_3(G, \mathbf{c}) \rceil,$$

as required.

So assume that G is not a 2-star. If there is a node v such that $|N(v)| \geq 4$, let S be a member of \mathcal{S} that contains v . Otherwise, let S be any member of \mathcal{S} . It is routine to check that $\chi_3(G, \mathbf{c} - \chi^S) \leq \lceil \eta_3(G, \mathbf{c}) \rceil - 1$. Again the theorem follows by induction.

Case 2. G does not fulfill condition (ii).

Let $\mathbf{a} \cdot \mathbf{x} \leq \beta$ be such that $\frac{1}{\beta}(\mathbf{a} \cdot \mathbf{c}) = \eta_3(G, \mathbf{c})$. Denote by P the nodes of G that do not belong to the support of \mathbf{a} . Our goal will be to reduce (G, \mathbf{c}) into two weighted graphs (G', \mathbf{c}') and (G'', \mathbf{c}'') and then color the first one using induction and the second one using Corollary 4.3.

$$\text{Let } \mathcal{S}' = \begin{cases} \{S \in \mathcal{S} : S \cap P \neq \emptyset \text{ or } |S| = 2\} & \text{if } \mathbf{a} \cdot \mathbf{x} \leq \beta \text{ is of type (3b),} \\ \{S \in \mathcal{S} : S \cap P = \emptyset\} & \text{otherwise.} \end{cases}$$

Let $p = \sum_{S \in \mathcal{S}'} y_S$. By Lemma 4.1 and by the definition of $\mathbf{a} \cdot \mathbf{x} \leq \beta$, there exists a node $v \in V(G)$ such that $v \in S$ for all $S \in \mathcal{S}'$. (When $\mathbf{a} \cdot \mathbf{x} \leq \beta$ is of type (3a) or (3b) v is a node with coefficient 2 or 3, respectively, and in the other cases it is any node from the set T of the support of \mathbf{a} .) Thus v is not adjacent to any other node that belongs to a member of \mathcal{S}' . Let G' be the graph induced by the nodes different from

v that belong to some member of \mathcal{S}' , and let \mathbf{c}' be the restriction of $\sum_{S \in \mathcal{S}'} y_S \chi^S$ to $V(G')$. Let G'' be the graph induced by the support of \mathbf{a} , and let \mathbf{c}'' be the restriction of $\mathbf{c} - \sum_{S \in \mathcal{S}'} y_S \chi^S - \lceil p \rceil \chi^{\{v\}}$ to $V(G'')$.

Suppose for the moment that both \mathbf{c}' and \mathbf{c}'' are integral (so $p = \lceil p \rceil$). Because $c(V(G)) \leq 3\eta_3(G, \mathbf{c})$, not every $S \in \mathcal{S}'$ is of size 3. Thus $c'(V(G')) < 2p$ and by Corollary 4.3, (G, \mathbf{c}') has a coloring with at most p 2-stable sets. Each one of these stable sets can be extended to include v . In addition, \mathbf{y} restricted to $\mathcal{S} \setminus \mathcal{S}'$ gives a coloring of (G'', \mathbf{c}'') of value $\eta_3(G, \mathbf{c}) - p$. Moreover, (G'', \mathbf{c}'') fulfills condition (ii); therefore, by Case 1, (G'', \mathbf{c}'') has a coloring of value $\lceil \eta_3(G, \mathbf{c}) - p \rceil$. Combining these two colorings gives a coloring of (G, \mathbf{c}) of value $\lceil \eta_3(G, \mathbf{c}) - p \rceil + p = \lceil \eta_3(G, \mathbf{c}) \rceil$, as required.

To conclude the case and the lemma, we now show that, if necessary, the coloring vector \mathbf{y} can be altered (while keeping \mathcal{S} unchanged) so that it is still an optimal fractional coloring and both \mathbf{c}' and \mathbf{c}'' are integral.

If \mathbf{c}' and \mathbf{c}'' are nonintegral, then $\mathbf{a} \cdot \mathbf{x} \leq \beta$ must be an inequality of the form $2x(v) + x(N(v)) + x(K) \leq 3$, where v is the special node identified earlier and K has exactly two elements u and w . This follows from Lemma 4.1 and the fact that $c(P)$ is integral. (The mentioned inequality is the only one for which there can be more than one color class of size less than 3 in the optimal fractional coloring of its support.) Note that since \mathbf{c} is integral, u and w index the only nonintegral components of \mathbf{c}' and \mathbf{c}'' . Let $p_u = \sum_{\{S \in \mathcal{S}': u \in S\}} y_S$ and $p_w = \sum_{\{S \in \mathcal{S}': w \in S\}} y_S$. By Lemma 4.1, any member of \mathcal{S} that contains v must also contain u or w , so $p_u + p_w = c(v)$ and $p = p_u + p_w$. So p is an integer and, by assumption, p_u and p_w are both not integers. Consider the set of nodes $Q \subseteq P$ that belong to some member of \mathcal{S}' together with $\{v, u\}$. Since $c(Q)$ is an integer, either $y_{\{u,v\}} \notin \mathbb{Z}$ or there exists a $z \in Q$ such that $y_{\{u,v,z\}} \notin \mathbb{Z}$. In the former case, if $y_{\{u,v\}} \notin \mathbb{Z}$, then since $c(P) = \sum_{\{S \in \mathcal{S}': |S|=3\}} y_S \in \mathbb{Z}$ we have that $y_{\{v,u\}} + y_{\{v,w\}} \in \mathbb{Z}$. Now decrease $y_{\{v,u\}}$ by $y_{\{v,u\}} - \lfloor y_{\{v,u\}} \rfloor$ and increase $y_{\{v,w\}}$ by the same amount. In the latter case, $\{v, w, z\}$ must also be in \mathcal{S}' since $c(z) \in \mathbb{Z}$ and $y_{\{v,u,z\}} \notin \mathbb{Z}$. Then decrease $y_{\{v,u,z\}}$ by $y_{\{v,u,z\}} - \lfloor y_{\{v,u,z\}} \rfloor$ and increase $y_{\{v,w,z\}}$ by the same amount. Repetition of this procedure yields the desired coloring. \square

Proof of Theorem 4.4. We prove the theorem by induction on $\lceil \eta_3(G, \mathbf{c}) \rceil$. If (G, \mathbf{c}) fulfills one of the conditions of Lemma 4.5, we are done. If not, then choose any color class S from an optimal fractional coloring of (G, \mathbf{c}) by 3-stable sets and, if necessary, add nodes until S is either maximal of size 3. If $\eta_3(G, \mathbf{c} - \chi^S) < \lceil \eta_3(G, \mathbf{c}) \rceil - 1$, then we can apply induction and find a coloring of $(G, \mathbf{c} - \chi^S)$ with $\lceil \eta_3(G, \mathbf{c} - \chi^S) \rceil + 1$ 3-stable sets. Otherwise, it must be that $(G, \mathbf{c} - \chi^S)$ fulfills condition (i) of Lemma 4.5 and the result follows. \square

Again, the upper bound of the theorem is tight, for if G is a complete bipartite graph where every color class is larger than 3 and equal to 1 mod 3, then $\chi_3(G) = \lceil \eta_3(G) \rceil + 1$.

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